Propositional Logic Intro

Note 1

Proposition: A statement with a truth value; it is either true or false.

Propositions can be combined to form more complicated expressions, using the following operations:

Operators		Quantifiers		Implication operations	
$\wedge$	and	$\forall$	for all	Implication	$P \implies Q$
$\vee$	or	$\exists$	there exists	Inverse	$\neg P \implies \neg Q$
$\neg$	not			Converse	$Q \implies P$
$\implies$	implies			Contrapositive	$\neg Q \implies \neg P$
$\equiv$	equivalent to				

Further, for an implication $P \implies Q$ where P is the *hypothesis* and Q is the *conclusion*, it is useful to know that $P \implies Q \equiv \neg P \vee Q$. Additionally, observe that any implication is logically equivalent to its contrapositive.

DeMorgan's Laws: The following identities can be helpful when simplifying expressions and distributing negations.

- $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$
- $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
- $\neg(\forall x\, P(x)) \equiv \exists x(\neg P(x))$
- $\neg(\exists x\, P(x)) \equiv \forall x(\neg P(x))$

1 Propositional Practice

Note 1

Convert the following English sentences into propositional logic and the following propositions into English. State whether or not each statement is true with brief justification.

Recall that $\mathbb{R}$ is the set of reals, $\mathbb{Q}$ is the set of rationals, $\mathbb{Z}$ is the set of integers, and $\mathbb{N}$ is the set of natural numbers. The notation "$a \mid b$", read as "a divides b", means that a is a divisor of b.

(a) There is a real number which is not rational.

(b) All integers are natural numbers or are negative, but not both.

(c) If a natural number is divisible by 6, it is divisible by 2 or it is divisible by 3.

(d) $\neg(\forall x \in \mathbb{Q})(x \in \mathbb{Z})$

(e) $(\forall x \in \mathbb{Z})(((2 \mid x) \vee (3 \mid x)) \implies (6 \mid x))$

(f) $(\forall x \in \mathbb{N})((x > 7) \implies ((\exists a, b \in \mathbb{N})\ (a + b = x)))$

2 Truth Tables

Note 1

Determine whether the following equivalences hold, by writing out truth tables. Clearly state whether or not each pair is equivalent.

(a) $P \wedge (Q \vee P) \equiv P \wedge Q$

(b) $(P \vee Q) \wedge R \equiv (P \wedge R) \vee (Q \wedge R)$

(c) $(P \wedge Q) \vee R \equiv (P \vee R) \wedge (Q \vee R)$

3 Implication

Note 0
Note 1

Which of the following implications are always true, regardless of P? Give a counterexample for each false assertion (i.e. come up with a statement $P(x,y)$ that would make the implication false).

(a) $\forall x \forall y P(x,y) \implies \forall y \forall x P(x,y)$.

(b) $\forall x \exists y P(x,y) \implies \exists y \forall x P(x,y)$.

(c) $\exists x \forall y P(x,y) \implies \forall y \exists x P(x,y)$.