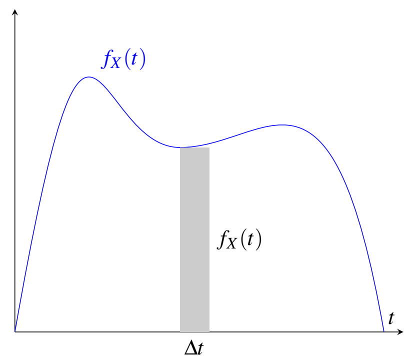
CS 70 Discrete Mathematics and Probability Theory
Spring 2025 Rao

DIS 13A

Continuous Probability Intro I

In discrete probability, we are only concerned with RVs that take on countably many values; now, in continuous probability, we are interested in RVs that take on *uncountably* many values. Here, the most important difference is that $\mathbb{P}[X = k] = 0$ for any k, but we can have $\mathbb{P}[X \in (a,b)] > 0$.

This gives the motivation for the **probability density function (PDF)**, denoted as $f_X(t)$.



Recall from physics that in 1D, density = mass/length; we define *probability density* similarly, as probability/length. In particular, this means that the area of the rectangle, $f_X(t)\Delta t$ (a product of density and length), is a probability. Generalizing this idea to find the area under the curve, we can now find probabilities from the PDF:

$$\mathbb{P}[a < X < b] = \int_a^b f_X(t)\,\mathrm{d}t.$$

From here, we define the **cumulative distribution function (CDF)** as

$$F_X(t) = \mathbb{P}[X < t] = \int_{-\infty}^t f_X(u)\,\mathrm{d}u.$$

From the fundamental theorem of calculus, we have $\frac{\mathrm{d}}{\mathrm{d}t}F_X(t) = f_X(t)$; the derivative of the CDF is the PDF.

Properties:

- $f_X(t) \geq 0$, and $\int_{-\infty}^{\infty} f_X(t)\,\mathrm{d}t = 1$
- $F_X(t)$ must be non-decreasing, with $\lim_{t\to\infty} F_X(t) = 1$ and $\lim_{t\to-\infty} F_X(t) = 0$

Other probability concepts follow naturally from these definitions; the only major difference from discrete probability is that sums turn into integrals, and $\mathbb{P}[X = t]$'s turn into $f_X(t)$'s. For example,

	Discrete	Continuous
Expectation	$\mathbb{E}[X] = \sum_t t \cdot \mathbb{P}[X = t]$	$\mathbb{E}[X] = \int_{-\infty}^{\infty} t \cdot f_X(t)\,\mathrm{d}t$
LOTUS	$\mathbb{E}[g(X)] = \sum_t g(t) \cdot \mathbb{P}[X = t]$	$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(t) \cdot f_X(t)\,\mathrm{d}t$
Total Probability	$\mathbb{P}[A] = \sum_{t=1}^n \mathbb{P}[A \mid X = t]\,\mathbb{P}[X = t]$	$\mathbb{P}[A] = \int_{-\infty}^{\infty} \mathbb{P}[A \mid X = t] f_X(t)\,\mathrm{d}t$

Exponential Distribution: $X \sim \text{Exponential}(\lambda)$, the continuous analog to the geometric distribution; it models the amount of time needed to wait until a success, where the rate of success is λ.

$$f_X(t) = \lambda e^{-\lambda t} \qquad \mathbb{E}[X] = \frac{1}{\lambda}$$

$$F_X(t) = 1 - e^{-\lambda t} \qquad \mathrm{Var}(X) = \frac{1}{\lambda^2}$$

Similar to the geometric distribution, the exponential distribution also has the memoryless property:

$$\mathbb{P}[X > m + n \mid X > m] = \mathbb{P}[X > n].$$

1 Continuous Intro

Note 21

(a) Is

$$g(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

a valid density function? Why or why not? Is it a valid CDF? Why or why not?

(b) Calculate the PDF $f_X(x)$, along with $\mathbb{E}[X]$ and $\mathrm{Var}(X)$ if the CDF of X is

$$F_X(x) = \begin{cases} 0, & x \le 0 \\ \frac{x}{\ell}, & 0 \le x \le \ell, \\ 1, & x \ge \ell \end{cases}$$

(c) Suppose X and Y are independent and have densities

$$f_X(x) = \begin{cases} 2x, & 0 \le x \le 1, \\ 0, & \text{otherwise}, \end{cases} \qquad f_Y(y) = \begin{cases} 1, & 0 \le y \le 1, \\ 0, & \text{otherwise}. \end{cases}$$

We will use these X, Y defined here for the rest of the subparts of this question. What is their joint density? (Hint: We can use independence in much the same way that we did in discrete probability)

(d) Calculate $\mathbb{E}[XY]$ for the X and Y in part (c).

(e) Recall the definition of the joint distribution of X and Y: $\mathbb{P}[a \le X \le b, c \le Y \le d] = \int_a^b \int_c^d f_{X,Y}(x,y)\,\mathrm{d}y\,\mathrm{d}x$. Derive the marginal density function for continuous random variables (Hint: Start by computing the CDF $F_X(x)$):

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}y$$

(f) Let $Z = X + Y$ for the X and Y in part (c). Derive an expression for the joint distribution $f_{X,Z}(x,z)$ in terms of $f_X(x), f_Y(y)$ and use this to compute (but not evaluate) an integral expression for $f_Z(z)$.

2 Darts Again

Note 21

Edward and Khalil are playing darts on a circular dartboard.

Edward's throws are uniformly distributed over the entire dartboard, which has a radius of 10 inches. Khalil has good aim (but his throws may land outside of the dartboard); the distance of his throws from the center of the dartboard follows an exponential distribution with parameter $\frac{1}{2}$.

Say that Edward and Khalil both throw one dart at the dartboard. Let X be the distance of Edward's dart from the center, and Y be the distance of Khalil's dart from the center of the dartboard. What is $\mathbb{P}[X < Y]$, the probability that Edward's throw is closer to the center of the board than Khalil's? Leave your answer in terms of an unevaluated integral.

[*Hint:* X is not uniform over $[0, 10]$. Solve for the distribution of X by first computing the CDF of X, $\mathbb{P}[X < x]$.]

3 Lunch Meeting

Note 21

Alice and Bob agree to try to meet for lunch between 12 PM and 1 PM at their favorite sushi restaurant. Being extremely busy, they are unable to specify their arrival times exactly, and can say only that each of them will arrive (independently) at a time that is uniformly distributed within the hour. In order to avoid wasting precious time, if the other person is not there when they arrive they agree to wait exactly fifteen minutes before leaving.

(a) Provide a sketch of the joint distribution of the arrival times of Alice and Bob. For which region of the graph will Alice and Bob actually meet?

(b) Based on your sketch, what is the probability that they will actually meet for lunch?