Continuous Probability Intro II

Normal (Gaussian) Distribution: $X \sim N(\mu, \sigma^2)$

The normal distribution occurs frequently in nature, mostly due to the Central Limit Theorem.

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$
$$F_X(x) = \Phi(x)$$

Note that there is no closed form expression for the CDF of the normal distribution.

Properties:

- A **standard normal** distribution is denoted as $Z \sim N(0,1)$
- If $X \sim N(\mu, \sigma^2)$, then $\frac{X-\mu}{\sigma} \sim N(0,1)$
- Generally, if $X \sim N(\mu, \sigma^2)$, then $aX + b \sim N(a\mu + b, a^2\sigma^2)$
- If $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$ are independent, then

$$aX + bY \sim N(a\mu_X + b\mu_Y, a^2\sigma_X^2 + b^2\sigma_Y^2).$$

Central Limit Theorem: Let $X_1, X_2, \ldots, X_n$ be i.i.d random variables with mean μ and variance σ^2, and let

$$S_n = \sum_{i=1}^{n} X_i \qquad A_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Note that

$$\mathbb{E}[A_n] = \mu \qquad \text{Var}(A_n) = \frac{\sigma^2}{n}$$

The central limit theorem states that as $n \to \infty$, $A_n \to N(\mu, \frac{\sigma^2}{n})$. Or,

$$S_n \to N(n\mu, n\sigma^2)$$
$$\frac{A_n - \mu}{\sigma/\sqrt{n}} \to N(0,1)$$
$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \to N(0,1)$$

These four equations are all equivalent formations of the same idea, which is that the sample mean (of i.i.d random variables) will converge to a normal distribution preserving the mean and variance of the sample mean, as $n \to \infty$ (and the same holds for the sample sum, and shifted/scaled versions of them).

1 Interesting Gaussians

Note 21

(a) If $X \sim N(0, \sigma_X^2)$ and $Y \sim N(0, \sigma_Y^2)$ are independent, then what is $\mathbb{E}\left[(X+Y)^k\right]$ for any *odd* $k \in \mathbb{N}$?

(b) Let $f_{\mu,\sigma}(x)$ be the density of a $N(\mu, \sigma^2)$ random variable, and let X be distributed according to $\alpha f_{\mu_1,\sigma_1}(x) + (1-\alpha) f_{\mu_2,\sigma_2}(x)$ for some $\alpha \in [0,1]$. Compute $\mathbb{E}[X]$ and $\mathrm{Var}(X)$. Is X normally distributed?

2 Binomial Concentration

Note 21

Here, we will prove that the binomial distribution is *concentrated* about its mean as the number of trials tends to ∞. Suppose we have i.i.d. trials, each with a probability of success $1/2$. Let S_n be the number of successes in the first n trials (n is a positive integer).

(a) Compute the mean and variance of S_n.

(b) How should we define Z_n in terms of S_n to ensure that Z_n has mean 0 and variance 1?

(c) What is the distribution of Z_n as $n \to \infty$?

(d) Use the bound $\mathbb{P}[Z > z] \le (\sqrt{2\pi}z)^{-1} \mathrm{e}^{-z^2/2}$ when Z is a standard normal in order to approximately bound $\mathbb{P}[S_n/n > 1/2 + \delta]$, where $\delta > 0$.

3 Erasures, Bounds, and Probabilities

Note 21

Alice is sending 1000 bits to Bob. The probability that a bit gets erased is p, and the erasure of each bit is independent of the others.

Alice is using a scheme that can tolerate up to one-fifth of the bits being erased. That is, as long as Bob receives at least 801 of the 1000 bits correctly, he can decode Alice's message.

In other words, Bob becomes unable to decode Alice's message only if 200 or more bits are erased. We call this a "communication breakdown", and we want the probability of a communication breakdown to be at most 10^{-6}.

(a) Use Chebyshev's inequality to upper bound p such that the probability of a communications breakdown is at most 10^{-6}.

(b) As the CLT would suggest, approximate the fraction of erasures by a Gaussian random variable (with suitable mean and variance). Use this to find an approximate bound for p such that the probability of a communications breakdown is at most 10^{-6}.

You may use that $\Phi^{-1}(1-10^{-6}) \approx 4.753$.