CS 70 Discrete Mathematics and Probability Theory
Spring 2025 Rao **Discussion 14B**

Markov Chains Intro II

Note 22

Recall that a Markov chain is defined with the following: the state space $\mathscr{X}$, the transition matrix P, and the initial distribution π_0. This implicitly defines a sequence of random variables X_n with distribution π_n, which denote the state of the Markov chain at timestep n. This sequence of random variables also obey the Markov property: the transition probabilities only depend on the current state, and not any prior states.

A before B: Suppose we want to compute the probability of reaching state A before reaching state B. To compute this quantity, let $\alpha(i) = \mathbb{P}[A \text{ before } B \mid \text{at } i]$. Then, we have:

$$\begin{aligned}\alpha(A) &= 1\\ \alpha(B) &= 0\\ \alpha(i) &= \sum_j P(i,j)\alpha(j)\end{aligned}$$

Here, we use the law of total probability when computing $\alpha(i)$; we consider all possible transitions *out of* state i. These are called the **first step equations (FSE)**.

Hitting time: Suppose we want to compute the expected number of steps until you reach state A. To compute this quantity, let $\beta(i) = \mathbb{E}[\text{steps until } A \mid \text{at } i]$. Then, the first step equations become:

$$\begin{aligned}\beta(A) &= 0\\ \beta(i) &= 1 + \sum_j P(i,j)\beta(j)\end{aligned}$$

Here, we use the law of total expectation when computing $\beta(i)$; we consider all possible transitions *out of* state i.

1 Skipping Stones

Note 22

We consider a simple Markov chain model for skipping stones on a river, but with a twist: instead of trying to make the stone travel as far as possible, you want the stone to hit a target. Let the set of states be $\mathscr{X} = \{1,2,3,4,5\}$. State 3 represents the target, while states 4 and 5 indicate that you have overshot your target. Assume that from states 1 and 2, the stone is equally likely to skip forward one, two, or three steps forward. If the stone starts from state 1, compute the probability of reaching our target before overshooting, i.e. the probability of $\{3\}$ before $\{4,5\}$.

2 Three Tails

Note 22

You flip a fair coin until you see three tails in a row. What is the average number of heads that you'll see until getting TTT?

Hint: It can help to start by thinking about how to compute the number of *coins* flipped until getting TTT, and then slightly modifying your equations to solve the original question.