CS 70	Discrete Mathematics and Probability Theory	
Spring 2025	Rao	HW 02

Due: Saturday, 2/8, 4:00 PM
Grace period until Saturday, 2/8, 6:00 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Universal Preference

Note 4

Suppose that preferences in a stable matching instance are universal: all n jobs share the preferences $C_1 > C_2 > \cdots > C_n$ and all candidates share the preferences $J_1 > J_2 > \cdots > J_n$.

(a) What pairing do we get from running the algorithm with jobs proposing? Prove that this happens for all n.

(b) What pairing do we get from running the algorithm with candidates proposing? Justify your answer.

(c) What does this tell us about the number of stable pairings? Justify your answer.

2 Pairing Up

Note 4

Prove that for every even $n \geq 2$, there exists an instance of the stable matching problem with n jobs and n candidates such that the instance has at least $2^{n/2}$ distinct stable matchings.

(*Hint*: It can help to start with some small examples; find an instance for $n = 2$, and think about how you can use these preference lists to construct an instance for $n = 4$. After this, you should be in a good position to generalize the construction for all even n.)

3 Upper Bound

Note 4

In the notes, we show that the stable matching algorithm terminates in at most n^2 days. Prove the following stronger result: the stable matching algorithm will always terminate in at most $(n-1)^2 + 1 = n^2 - 2n + 2$ days.

4 Short Tree Proofs

Note 5 Let $G = (V, E)$ be an undirected graph with $|V| \geq 1$.

(a) Prove that every connected component in an acyclic graph is a tree.

(b) Suppose G has k connected components. Prove that if G is acyclic, then $|E| = |V| - k$.

(c) Prove that a graph with $|V|$ edges contains a cycle.

5 Proofs in Graphs

Note 5 (a) Suppose California has n cities ($n \geq 2$) such that for every pair of cities X and Y, either X has a road to Y or Y has a road to X. Further, suppose that all roads are one-way streets.

Prove that regardless of the configuration of roads, there always exists a city which is reachable from every other city by traveling through at most 2 roads.

[*Hint:* Induction]

(b) Consider a connected graph G with n vertices which has exactly $2m$ vertices of odd degree, where $m > 0$. Prove that there are m walks that *together* cover all the edges of G (i.e., each edge of G occurs in exactly one of the m walks, and each of the walks should not contain any particular edge more than once).

[*Hint:* In lecture, we have shown that a connected undirected graph has an Eulerian tour if and only if every vertex has even degree. This fact may be useful in the proof.]

(c) Prove that any graph G is bipartite if and only if it has no tours of odd length.

[*Hint:* In one of the directions, consider the lengths of paths starting from a given vertex.]