# Today.

Finish up counting.

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Countabiity.

# Some Practice.

How many orderings of letters of CAT?

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total orderings of 7 letters.

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total "extra counts" or orderings of three A's?

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4 S's, 4 I's, 2 P's.

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11 letters total.

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**First rule:** $n_1 \times n_2 \cdots \times n_3$.

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$\binom{n}{k} = \frac{n!}{(n-k)!k!}$. "$n$ choose $k$"

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Sample $k$ times $n$ with replacement and order doesn't matter:
$\binom{k+n-1}{n-1}$.

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Sample $k$ items out of $n$

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Order does not matter:

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- Without replacement:
  - Order matters: $n \times n-1 \times n-2 \ldots \times n-k+1 = \frac{n!}{(n-k)!}$
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Order matters: $n \times n$

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# Splitting 5 dollars..

How many ways can Alice, Bob, and Eve split 5 dollars.

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Separate Alice's dollars from Bob's and then Bob's from Eve's.

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Five dollars are five stars: $\star\star\star\star\star$.

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Alice: 2, Bob: 1, Eve: 2.

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# Splitting 5 dollars..

How many ways can Alice, Bob, and Eve split 5 dollars.

Alice gets 3, Bob gets 1, Eve gets 1: $(A, A, A, B, E)$.

Separate Alice's dollars from Bob's and then Bob's from Eve's.

Five dollars are five stars: $\star\star\star\star\star$.

Alice: 2, Bob: 1, Eve: 2.
Stars and Bars: $\star\star|\star|\star\star$.

Alice: 0, Bob: 1, Eve: 4.
Stars and Bars: $|\star|\star\star\star\star$.

Each split "is" a sequence of stars and bars.
Each sequence of stars and bars "is" a split.

**Counting Rule: if there is a one-to-one mapping between two sets they have the same size!**

# Stars and Bars.

How many different 5 star and 2 bar diagrams?

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$\binom{7}{2}$ ways to split 5 dollars among 3 people.

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Or: $k$ unordered choices from set of $n$ possibilities with replacement.
**Sample with replacement where order doesn't matter.**

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$|S| = |T| = \binom{7}{2}$.

# Stars and Bars Poll

**Mark whats correct.**

(A) ways to split n dollars among k: $\binom{n+k-1}{k-1}$

(B) ways to split k dollars among n: $\binom{k+n-1}{n-1}$

(C) ways to split 5 dollars among 3: $\binom{7}{5}$

(D) ways to split 5 dollars among 3: $\binom{5+3-1}{3-1}$

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All correct.

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Two indistinguishable jokers in 54 card deck.
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How many 5 card poker hands? Choose 4 cards plus one of 2 jokers!

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# Pascal's Triangle

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0

1 1

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| | | | | |
|---|---|---|---|---|
| | | 0 | | |
| | 1 | | 1 | |
| 1 | | 2 | | 1 |

# Pascal's Triangle

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | 0 | | | |
| | | 1 | | 1 | | |
| | 1 | | 2 | | 1 | |
| 1 | | 3 | | 3 | | 1 |

# Pascal's Triangle

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | 0 | | | | |
| | | | 1 | | 1 | | | |
| | | 1 | | 2 | | 1 | | |
| | 1 | | 3 | | 3 | | 1 | |
| 1 | | 4 | | 6 | | 4 | | 1 |

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| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | 0 | | | | |
| | | | 1 | | 1 | | | |
| | | 1 | | 2 | | 1 | | |
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# Pascal's Triangle

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

Row $n$: coefficients of $(a+b)^n = (a+b)(a+b)\cdots(a+b)$.

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| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | 0 | | | | |
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Foil (4 terms)

# Pascal's Triangle

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Row $n$: coefficients of $(a+b)^n = (a+b)(a+b)\cdots(a+b)$.

Foil (4 terms) on steroids:

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Pascal's rule $\Longrightarrow$ $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

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So, $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$. $\square$

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# Simple Inclusion/Exclusion

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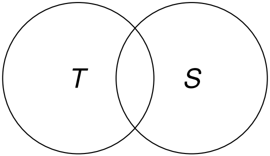

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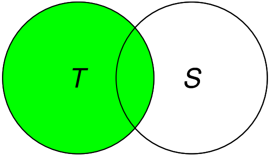


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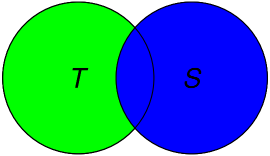


In $T$. $\implies$ $|T|$
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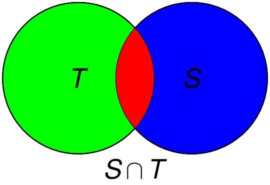


In $T$. $\implies$ $|T|$
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Elements in $S \cap T$ are counted twice.

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by adding number of subsets of size 1, 2, 3,...

Also reasoned about subsets that contained
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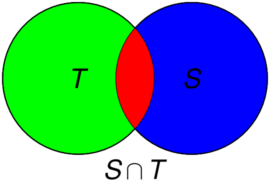


In $T$. $\implies |T|$
In $S$. $\implies + \ |S|$
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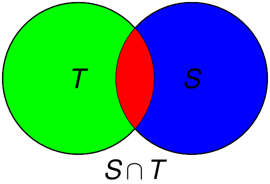


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Answer: $|S| + |T| - |S \cap T| = 10^9 + 10^9 - 10^8$.

# Inclusion/Exclusion

$|A_1 \cup \cdots \cup A_n| =$
$\sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots A_n|$.

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Total: 3 - 3 + 1 = 1.

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Formulaically: $x$ is in intersection of three sets.

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$|A_1 \cup \cdots \cup A_n| =$
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  Element $x$ in $m$ sets: $x \in A_{i_1} \cap A_{i_2} \cdots \cap A_{i_m}$.

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Counted $\binom{m}{i}$ times in $i$th summation.

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Binomial Theorem:
$(x+y)^m = \binom{m}{0} x^m + \binom{m}{1} x^{m-1} y + \binom{m}{2} x^{m-2} y^2 + \cdots \binom{m}{m} y^m$.

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- Proof: $m$ factors in product: $(x+y)(x+y)\cdots(x+y)$.

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Proof: $m$ factors in product: $(x+y)(x+y)\cdots(x+y)$.

Get a term $x^{m-i} y^i$ by choosing $i$ factors to use for $y$.

# Inclusion/Exclusion

$|A_1 \cup \cdots \cup A_n| =$
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Propositional forms use $\wedge, \vee, \neg$.

Propositional forms correspond to truth tables.

Logical equivalence of forms means same truth tables.

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$Q(n)$ could be true on evens and $P(n)$ could be true on false.
The left hand side is true, and the right is false.

# Lecture 2

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Hole anywhere on tiles.

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Induction $\equiv$ Recursion.

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What's after 0,

# Poll:lecture 3.

What's after 0, 1,

# Poll:lecture 3.

What's after 0, 1, 2,

# Poll:lecture 3.

What's after 0, 1, 2, 3, ...

# Poll:lecture 3.

What's after 0, 1, 2, 3, ...

What's before 100,

# Poll:lecture 3.

What's after 0, 1, 2, 3, ...

What's before 100, 99,

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What's after 0, 1, 2, 3, ...

What's before 100, 99, 98,

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Program works on smaller inputs.

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  ARGUMENT.

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ARGUMENT.
So it works on this one.

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WriteAsPrimes(x):

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WriteAsPrimes(x):
if prime(x): write x

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write(a), write(b).

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Analysis of cool algorithm with interesting goal: stability.

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Proofs carefully use definition:
- Stability:
  - Improvement Lemma plus every day the job gets to choose.
- Optimality proof:
  - Job Optimality:
    - contradiction of the existence of a better *stable* pairing.
      - that is, no rogue couple by improvement, job choice,
        and well ordering principle.
  - Candidate Pessimality:
    - contradiction plus cuz job optimality implies better pairing.

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Life Lesson: ask, you will do better even if rejection is hard.

# Exercise.

Why does it get better for candidates?

# Exercise.

Why does it get better for candidates?

They get to choose.

# Lecture 5 Summary.

Graphs.

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Graphs.
  Basics.

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Degree, Incidence, Sum of degrees is $2|E|$. Connectivity.

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Trees: degree 1 lemma $\implies$ equivalence of several definitions.

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maximal set of vertices that are connected.
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Trees: degree 1 lemma $\implies$ equivalence of several definitions.
$G$: $n$ vertices and $n-1$ edges and connected.
remove degree 1 vertex.
$n-1$ vertices, $n-2$ edges and connected $\implies$ acyclic.
(Ind. Hyp.)
degree 1 vertex is not in a cycle.
$G$ is acyclic.

# Exercise

Removing a degree 1 vertex does change connectivity of graph.

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Why?

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Why?

No path goes through it.

# Lecture 6 Summary.

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Tree. Plus adding edge adds face.

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  Replace $f$ in Euler.

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  degree $d$ vertex can be colored if $d+1$ colors.
  Small degree vertex in planar graph: 6 color theorem.
  Recolor separate and planarity: 5 color theorem.

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  Small degree vertex in planar graph: 6 color theorem.
  Recolor separate and planarity: 5 color theorem.
Graphs:
  Trees: sparsest connected.
  Complete:densest
  Hypercube:
    very connected, beautiful structure, bits,bits,bits.

# Exercise

Why $v + f = e + 2$?

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Why $v + f = e + 2$?

Tree had $e = v - 1$ and $f = 1$.

# Exercise

Why $v + f = e + 2$?

Tree had $e = v - 1$ and $f = 1$.
Adding edge makes new face.

# Lecture 7: Modular Arithmetic Lecture in a minute.

Modular Arithmetic: $x \equiv y \pmod{N}$ if $x = y + kN$ for some integer $k$.

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Division?

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Division? Multiply by multiplicative inverse.

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then $(x - y)$ must contain all primes in prime factorization of $N$,

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With induction.

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Any number can be written as a unique prime factorization.

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$gcd(n,m) = 1$. $n|x$ and $m|x$, implies $mn|x$.

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Unique prime factorization of any natural number.

Claim: if $p|n$ and $n = xy$, $p|x$ of $p|x$.

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Product of elts == for range/domain: $a^{p-1}$ factor in range.

# Exercise

Unique? $x = a \pmod{m}, x = b \pmod{n}$.

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Good for Encryption and Signature Schemes.

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$p | x^p - x$

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$P(x) = Q(x)$ on $d+1$ points $\implies P(x) = Q(x)$.

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Secret Sharing:
$k$ points on degree $k-1$ polynomial is great!
Can hand out $n$ points on polynomial as shares.

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Unique polynomial $P(x)$ that goes through $d+1$ points?

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$P(x) - Q(x)$ can only have $d$ roots.

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# Lecture 12/13.

First Rule of counting:

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$n_i$ possibilitities for $i$th choice :

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Count with order:

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Disjoint – so add!