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Disjoint – so add!

Poll: How big is infinity?

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Mark what's true.
(A) There are more real numbers than natural numbers.
(B) There are more rational numbers than natural numbers.
(C) There are more integers than natural numbers.
(D) pairs of natural numbers $>>$ natural numbers.

Same Size. Poll.

Two sets are the same size?

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(A) Bijection between the sets.
(B) Count the objects and get the same number $\implies$ same size.
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0,

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How to count?

0, 1,

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How to count?

0, 1, 2,

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How to count?

0, 1, 2, 3,

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How to count?

0, 1, 2, 3, ...

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How to count?

0, 1, 2, 3, . . .

The Counting numbers.

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0, 1, 2, 3, $\ldots$

The Counting numbers.
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S is **countable** if there is a bijection between S and some subset of $\mathbb{N}$.

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Enumerating a set implies countable.
Corollary: Any subset T of a countable set S is countable.

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All countably infinite sets have the same cardinality.

Enumeration example.

All binary strings.

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Never get to 1.

More fractions?

Enumerate the rational numbers in order...

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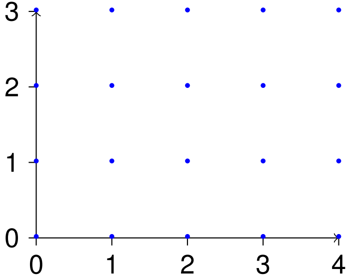

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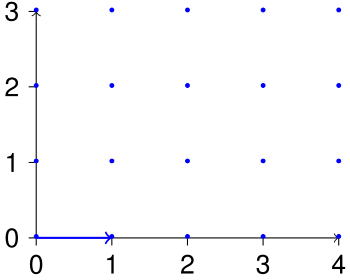

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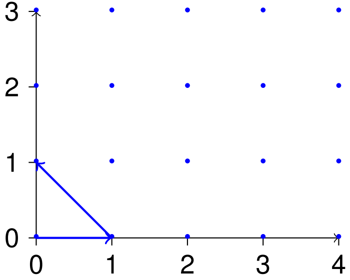

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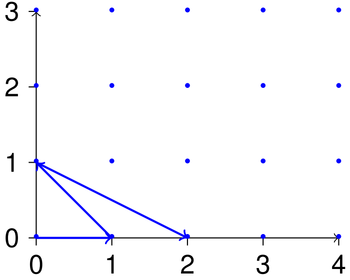

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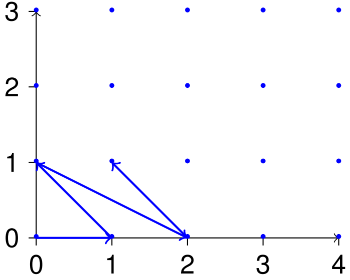

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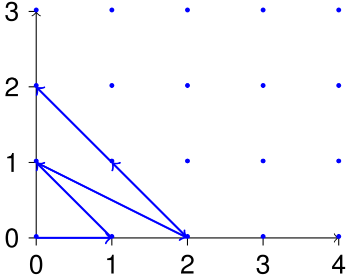

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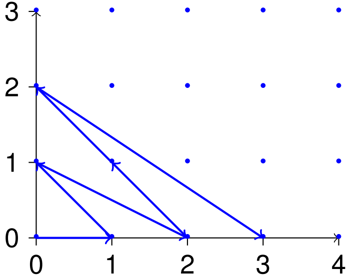

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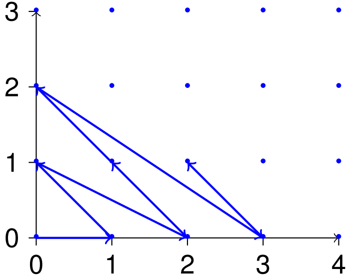
$(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),\ldots\ldots$



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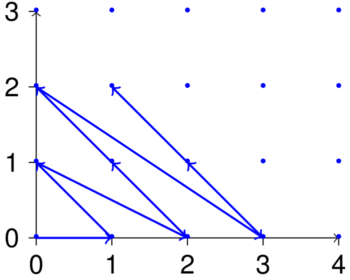

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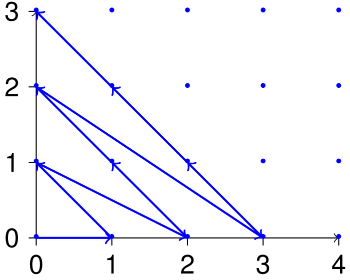

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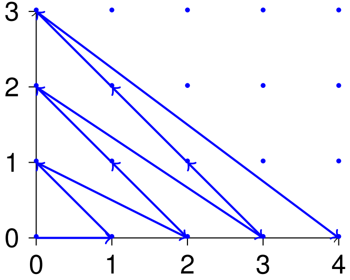

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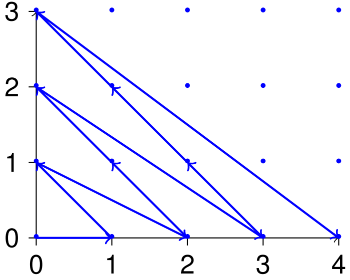

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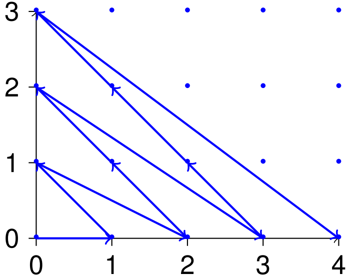

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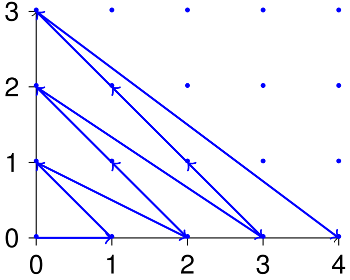

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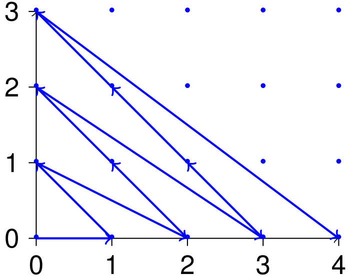

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Same size as the natural numbers!!

Poll.

Enumeration to get bijection with naturals?

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(A) Integers: First all negatives, then positives.

Poll.

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(B) Integers: By absolute value, break ties however.

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(B),(C), (F).

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Real numbers..

Real numbers are same size as integers?

The reals.

Are the set of reals countable?

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Construct "diagonal" number: .7

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⋮

Construct “diagonal” number: .77

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⋮

Construct “diagonal” number: .7767

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Subset $[0,1]$ is not countable!!

All reals?

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What about all reals?

All reals?

Subset $[0, 1]$ is not countable!!

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No.

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Any subset of a countable set is countable.

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No.

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If reals are countable then so is $[0, 1]$.

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1. Assume that a set S can be enumerated.

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Another diagonalization.

The set of all subsets of N.

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Example subsets of N: $\{0\}$,

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If ith set in L does not contain i, $i \in D$.
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D is different from ith set in L for every i.
$\implies$ D is not in the listing.

D is a subset of N.

L does not contain all subsets of N.

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$, $\{0, \ldots, 7\}$,
evens, odds, primes,

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(The set of all subsets of S, is the **powerset** of N.)

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Mark parts of proof.

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(B), (C)?, (D), (E)

The Continuum hypothesis.

There is no set with cardinality between the naturals and the reals.

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First of Hilbert's problems!

Cardinalities of uncountable sets?

Cardinality of $[0, 1]$ smaller than all the reals?

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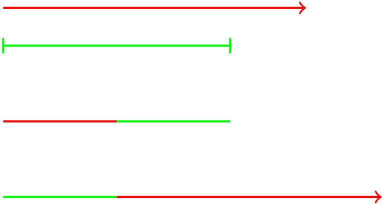

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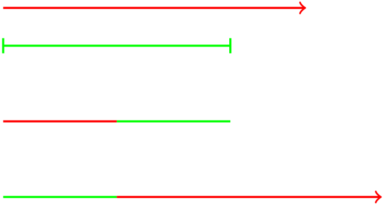

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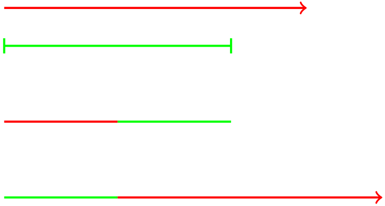

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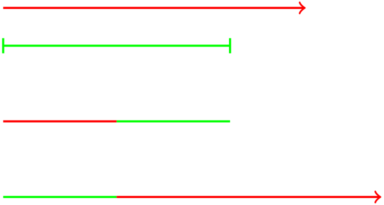

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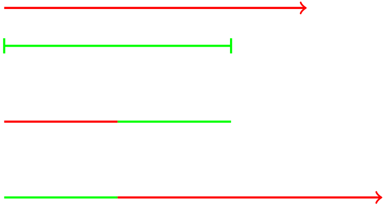

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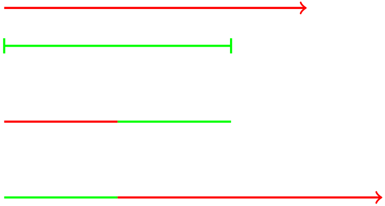

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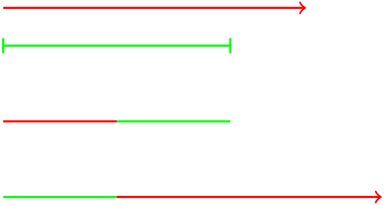

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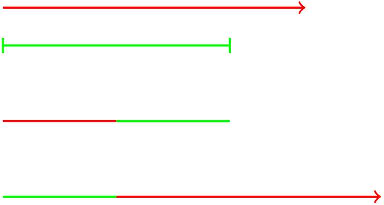

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$[0,1]$ is same cardinality as nonnegative reals!

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Generalized Continuum hypothesis.

There is no infinite set whose cardinality is between the cardinality of an infinite set and its power set.

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The powerset of a set is the set of all subsets.

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Uh oh....

The Barber!

The barber shaves (only) those who do not shave themselves.
Who shaves the Barber?

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(A) Mark shaves the Barber.

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(C) Mark is not Barber.
(D) Barber does not shave someone who does not shave themself.

The Barber!

The barber shaves (only) those who do not shave themselves.
Who shaves the Barber?

(A) Mark shaves the Barber.
(B) If Mark is Barber, Barber shaves themself.
(C) Mark is not Barber.
(D) Barber does not shave someone who does not shave themself.

Its a problem.

Generalized Continuum hypothesis.

There is no infinite set whose cardinality is between the cardinality of an infinite set and its power set.

Generalized Continuum hypothesis.

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The powerset of a set is the set of all subsets.

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There is no infinite set whose cardinality is between the cardinality of an infinite set and its power set.

The powerset of a set is the set of all subsets.

Recall: powerset of the naturals is not countable.

Resolution of hypothesis?

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Gödel. 1940.
Can't use math!

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If math doesn’t contain a contradiction.

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Is the statement above true?

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Uh oh....

The reals.

Are the set of reals countable?

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Lets consider the reals $[0, 1]$.

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Each real has a decimal representation.

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Diagonalization.

If countable, there a listing, L contains all reals.

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Construct "diagonal" number:

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Construct “diagonal” number: .7

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4: .345212312...
⋮

Construct “diagonal” number: .77

Diagonalization.

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1: .785398162...
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Construct "diagonal" number: .7767

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Diagonal Number: Digit i is 7 if number i's ith digit is not 7

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Contradiction!

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Contradiction!

Subset $[0, 1]$ is not countable!!

All reals?

Subset $[0, 1]$ is not countable!!

All reals?

Subset $[0, 1]$ is not countable!!

What about all reals?

All reals?

Subset $[0, 1]$ is not countable!!

What about all reals?
No.

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Any subset of a countable set is countable.

All reals?

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What about all reals?
No.

Any subset of a countable set is countable.

If reals are countable then so is $[0, 1]$.

Diagonalization.

1. Assume that a set S can be enumerated.

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5. Show that t is in S.
6. Contradiction.

Another diagonalization.

The set of all subsets of N.

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Example subsets of N: $\{0\}$,

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$, $\{0, \ldots, 7\}$,

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Example subsets of N: $\{0\}$, $\{0, \ldots, 7\}$,
evens, odds,

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Assume is countable.

Another diagonalization.

The set of all subsets of *N*.

Example subsets of *N*: $\{0\}$, $\{0, \ldots, 7\}$,
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Assume is countable.

There is a listing, *L*, that contains all subsets of *N*.

Another diagonalization.

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There is a listing, L, that contains all subsets of N.

Define a diagonal set, D:

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If ith set in L does not contain i, $i \in D$.

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D is different from ith set in L for every i.

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Theorem: The set of all subsets of N is not countable.

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L does not contain all subsets of N.

Contradiction.

Theorem: The set of all subsets of N is not countable.
(The set of all subsets of S, is the **powerset** of N.)

Poll: diagonalization Proof.

Mark parts of proof.

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Mark parts of proof.

(A) Integers are larger than naturals cuz obviously.
(B) Integers are countable cuz, interleaving bijection.
(C) Reals are uncountable cuz obviously!
(D) Reals can't be in a list: diagonal number not on list.
(E) Powerset in list: diagonal set not in list.

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(B), (C)?, (D), (E)

Rao is freaked out.

Are real numbers even real?

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Almost all real numbers can't be described.

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π?

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The ratio of the perimeter of a circle to its diameter.

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$\lim_{n\to\infty}\sum_{i=0}^{n}\frac{(b-a)}{n}f(x_i)$, where $x_i = a + i\times(b-a)/n$.

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for a beautiful, succint, powerful idea.

What's the idea?

Rao is freaked out.

Are real numbers even real?

Almost all real numbers can't be described.

π?
The ratio of the perimeter of a circle to its diameter.

e? Transcendental number.
$\lim_{n\to\infty}(1+1/n)^n$.

$\sqrt{2}$? Algebraic number.
A solution of $x^2 = 2$.

Really, rationals seem fine for... say... calculus.

$\lim_{n\to\infty}\sum_{i=0}^{n}\frac{(b-a)}{n}f(x_i)$, where $x_i = a + i\times(b-a)/n$.

So why real numbers?

$\int_a^b f(x)dx$ is beautiful, succint notation
for a beautiful, succint, powerful idea.

What's the idea? Area.

Generalized Continuum hypothesis.

There is no infinite set whose cardinality is between the cardinality of an infinite set and its power set.

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The powerset of a set is the set of all subsets.

Resolution of hypothesis?

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Gödel. 1940.
Can't use math!

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If math doesn't contain a contradiction.

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(A) Barber not Mark. Barber shaves Mark.
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(C) Barber doesn’t shave themself.
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Its all true.

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Its all true. It's a problem.

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Recall: powerset of the naturals is not countable.

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Any set of axioms is either

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See Logicomix by Doxiaidis, Papadimitriou (was professor here), Papadatos, Di Donna.

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Write me a program checker!

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Check that the compiler works!

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How about.. Check that the compiler terminates on a certain input.

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Run P on I and check!

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Determines if $P(I)$ (P run on I) halts or loops forever.

Run P on I and check!

How long do you wait?

Something about infinity here, maybe?

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What is he talking about?

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(A) He is confused.
(B) Diagonalization.
(C) Welch-Berlekamp
(D) Professor is just strange.

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(B) and (D) maybe? and maybe (A).

Professor does me some love Welch-Berlekamp though!

Halt and Turing.

Proof:

Halt and Turing.

Proof: Assume there is a program *HALT*$(\cdot,\cdot)$.

Halt and Turing.

Proof: Assume there is a program $HALT(\cdot, \cdot)$.

Turing(P)

Halt and Turing.

Proof: Assume there is a program $HALT(\cdot, \cdot)$.

Turing(P)
1. If HALT(P,P) ="halts", then go into an infinite loop.

Halt and Turing.

Proof: Assume there is a program $HALT(\cdot, \cdot)$.

Turing(P)

1. If HALT(P,P) ="halts", then go into an infinite loop.
2. Otherwise, halt immediately.

Halt and Turing.

Proof: Assume there is a program $HALT(\cdot, \cdot)$.

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There is text that "is" the program HALT.

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Does Turing(Turing) halt?

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Can run Turing on Turing!

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Another view of proof: diagonalization.

Any program is a fixed length string.

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P_2	L	L	H	$\cdots$
P_3	L	H	H	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

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Programs?

What are programs?

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(A) Instructions.
(B) Text.
(C) Binary String.
(D) They run on computers.

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All are correct.

Proof play by play.

Assumed $\text{HALT}(P, I)$ existed.

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We are so smart!

Wow, that was easy!

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We should be famous!

No computers for Turing!

In Turing’s time.

No computers for Turing!

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No computers.

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No computers.

Adding machines.

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Adding machines.
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Concept of program as data wasn't really there.

Turing machine.

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A Turing machine.
– an (infinite) tape with characters

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– be in a state, and read a character

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Turing: AI, self modifying code, learning...

Turing and computing.

Just a mathematician?

Turing and computing.

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Turing and computing.

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Simulated the program by hand to play chess.

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  The polish machine...the *bomba.*

Computing on top of computing...

Computer, assembly code, programming language, browser, html, javascript..

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We can't get enough of building more Turing machines.

Undecidable problems.

Does a program, P, print “Hello World”?

Undecidable problems.

Does a program, P, print “Hello World”?
How?

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Does a program, P, print "Hello World"?
How? What is P?

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Undecidable problems.

Does a program, P, print "Hello World"?
How? What is P? Text!!!!!!

Find exit points and add statement: **Print** "Hello World."

Can a set of notched tiles tile the infinite plane?
Proof: simulate a computer. Halts if finite.

Does a set of integer equations have a solution?
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Undecidability for Diophantine set of equations
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Turing: personal.

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- British Government apologized (2009) and pardoned (2013).

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Program is text, so we can pass it to itself,
or refer to self.

Summary: decidability.

Computer Programs are an interesting thing.

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Computation is a lens for other action in the world.

Kolmogorov Complexity, Google, and CS70

Of strings, s.

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Minimum sized program that prints string s.

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for $i = 1$ to n: print '1'.

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What is the minimum I need to know (remember) to know stuff.

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What is the minimum I need to know (remember) to know stuff.

Radius of the earth?

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Calculus:

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Kolmorogorov Complexity View(s):
Continuous Interest Rate: $(1+r/n)^n \to e^r$.
Solution to: $dy/dx = y$,
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Calculus: what is minimum you need to know?
Depends on your skills!

Kolmogorov Complexity, Google, and CS70

What is the minimum I need to know (remember) to know stuff.

Radius of the earth? Distance to the sun? Population of the US? Acceleration due to gravity on earth?
Google. Plus reference.

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Depends on your skills! Conceptualization.
Reason and understand an argument and you can generate a lot.

Calculus

What is the first half of calculus about?

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The slope of a tangent line to a function at a point.

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Idea: use rise in function value!

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used foil.

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Perhaps.

Derivative of sine?

$\sin(x)$.

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What is x? An angle in radians.

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Let's call it θ and do derivative of $\sin\theta$.

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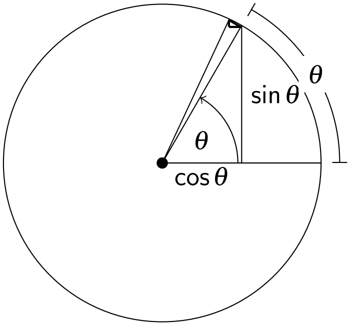

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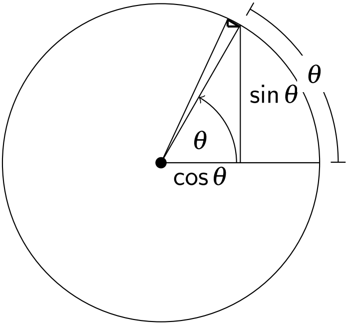

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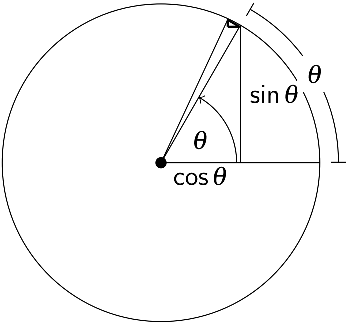

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Similar triangle!!!

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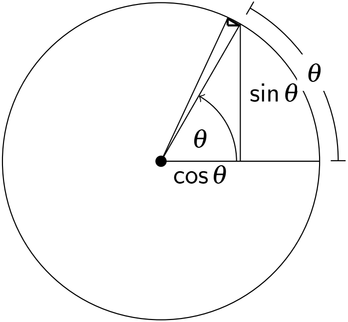

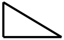

Rise.
Similar triangle!!!
Rise proportional to cosine!

Fundamental Theorem of Calculus.

Conceptual: Height times Width = Area.

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Useful?

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Useful?
 Speed times Time is Distance.

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If you change width, change in area is proportional to height.

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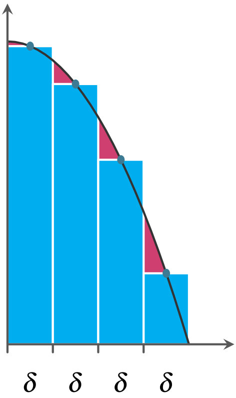
Derivative (rate of change) of Area (Integral) under curve, is height of curve.

Calculus



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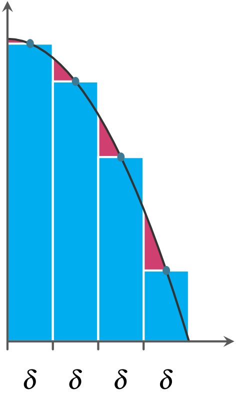
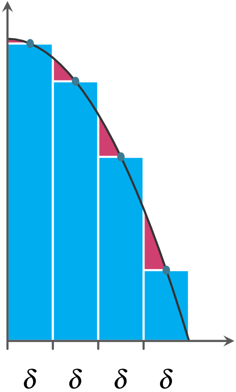


Riemann Sum/Integral: $\int_a^b f(x)dx = lim_{\delta \to 0} \sum_i \delta f(a_i)$

Calculus



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"Area is defined as rectangles and add up some thin ones."

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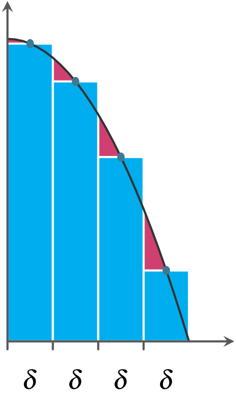

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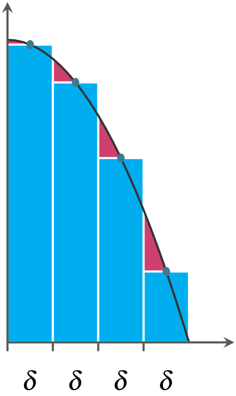

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"Rise over run of close together points."

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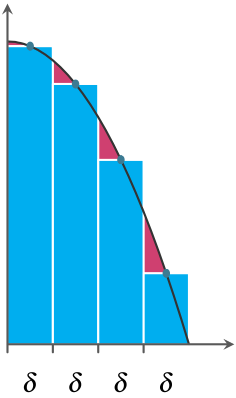

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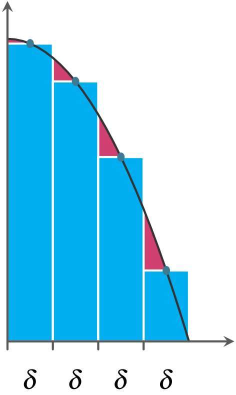

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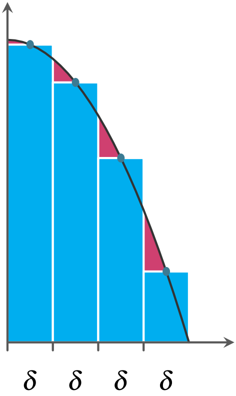

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"Area ($F(\cdot)$) under $f(x)$ grows at x, $F'(x)$, by $f(x)$"
Thus $F'(x) = f(x)$.

Arguments, reasoning.

What you know: slope, limit.

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Plus: definition.

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Knowing how to program

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Knowing how to program plus some syntax (google) gives the ability to program.

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...plus reasoning.

CS 70 : ideas.

Induction

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Induction $\equiv$ every integer has a next one.

CS 70 : ideas.

Induction $\equiv$ every integer has a next one. Graph theory.

- Number of edges is sum of degrees.
- $\Delta + 1$ coloring. Neighbors only take up Δ.
- Connectivity plus connected components.
- Eulerian paths: if you enter you can leave.
- Euler's formula: tree has $v - 1$ edges and 1 face plus
  - each extra edge makes additional face.
  - $v - 1 + (f - 1) = e$

CS 70 : ideas.

Number theory.

A divisor of x and y divides $x - y$.

The remainder is always smaller than the divisor.

$\implies$ Euclid's GCD algorithm.

Multiplicative Inverse.

Fermat's theorem from function with inverse is a bijection.

Gives RSA.

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Error Correction.
(Any) Two points determine a line.
(well, and d points determine a degree $d+1$-polynomials.
Cuz, factoring.
Find line by linear equations.
If a couple are wrong, then multiply them by zero, i.e., Error polynomial.

CS70 and your future?

What’s going on?

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Define. Understand properties. And build from there.

CS70 and your future?

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Define. Understand properties. And build from there.

Tools: reasoning, proofs, care.

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....and you will pursue probability in this course.