Lecture 17: Continuing Probability.

Events, Conditional Probability, Independence, Bayes' Rule

Probability Basics:Poll

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(A) A set and a function on the elements.
(B) The values of the function are real numbers.
(C) The values of the function are positive integers.
(D) An element of the set is an outcome.
(E) There is an experiment associated with a probability space.
(F) The values in the set are integers.

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(A),(B), (D), (E).

Probability Basics Review

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    1. $0 \leq Pr[\omega] \leq 1$.
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  - **Events.**
    Event $A \subseteq \Omega$, $Pr[A] = \sum_{\omega \in \Omega} Pr[\omega]$.

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A, B, C, D

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(b) Either induction, or argue over sample points.

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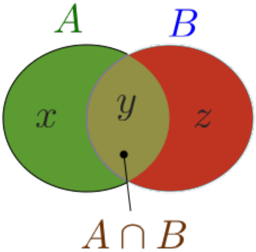

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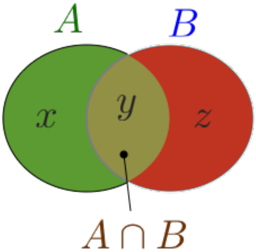

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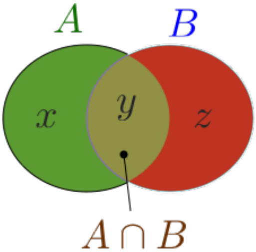

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Another view. Any $\omega \in A \cup B$ is in $A \cap \overline{B}$, $A \cup B$, or $\overline{A} \cap B$. So, add it up.

Roll a Red and a Blue Die.

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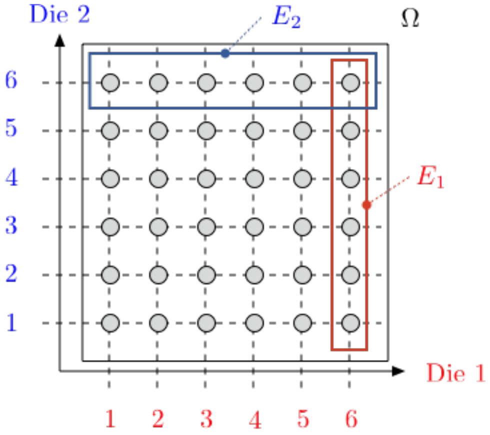

$$|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$$

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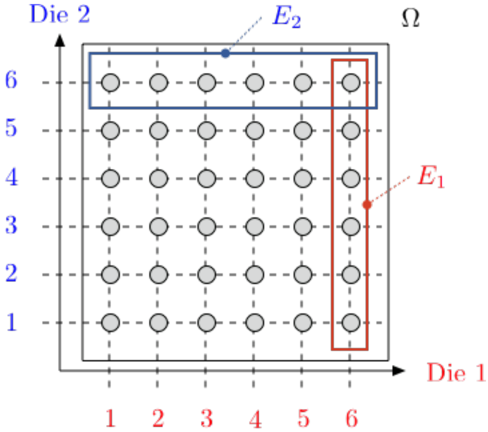

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$E_1 =$ 'Red die shows 6';

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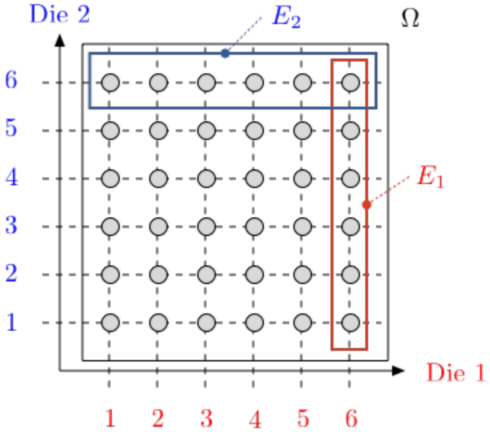

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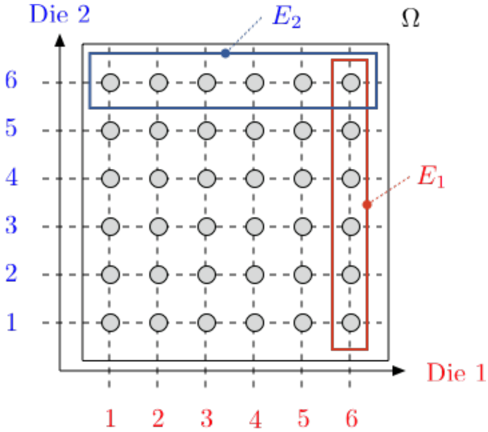

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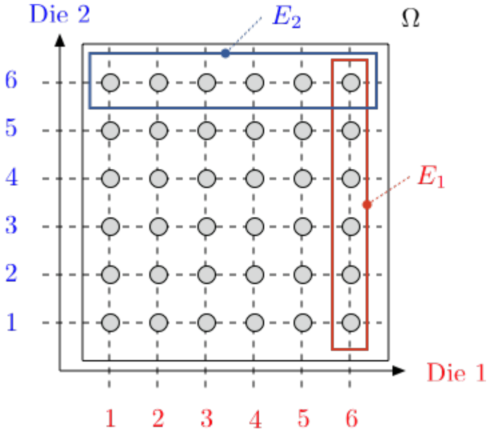

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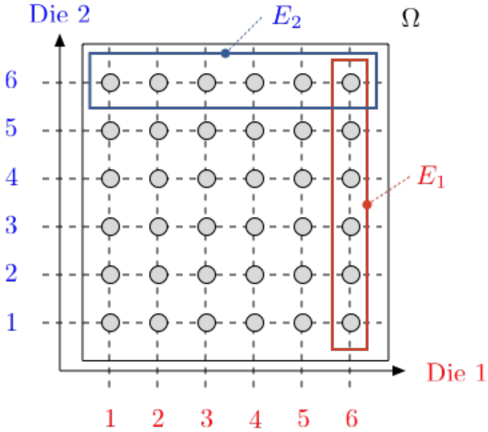

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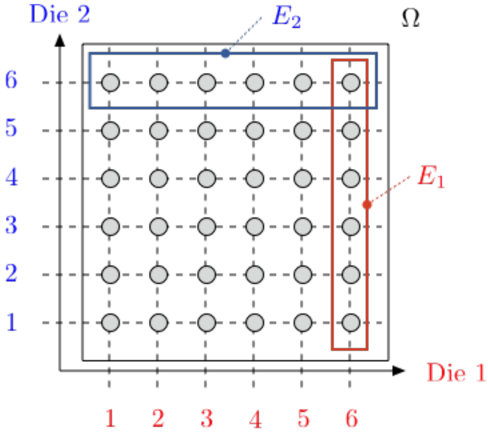

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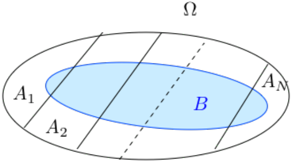
$$Pr[E_1] = \frac{6}{36}, Pr[E_2] = \frac{6}{36}, Pr[E_1 \cup E_2] = \frac{11}{36}.$$

Total probability

Assume that Ω is the union of the disjoint sets $A_1, \ldots, A_N$.



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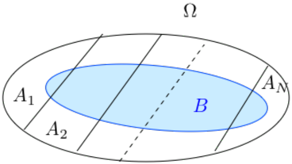

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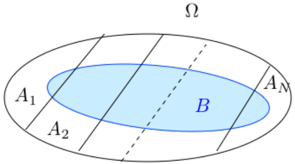

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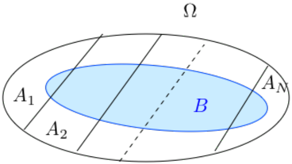

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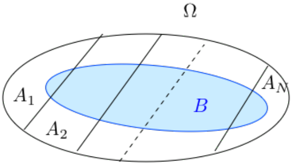

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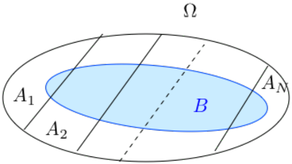

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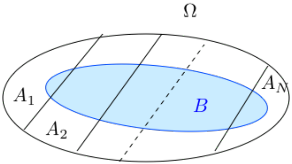
Total probability

Assume that Ω is the union of the disjoint sets $A_1, \ldots, A_N$.



Then,

$$Pr[B] = Pr[A_1 \cap B] + \cdots + Pr[A_N \cap B].$$

Indeed, B is the union of the disjoint sets $A_n \cap B$ for $n = 1, \ldots, N$.

In "math": $\omega \in B$ is in exactly one of $A_i \cap B$.

Adding up probability of them, get $Pr[\omega]$ in sum.

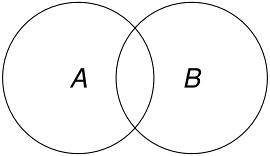
..Did I say...

Add it up.

Conditional Probability.

Definition: The **conditional probability** of *B* given *A* is

$$Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]}$$



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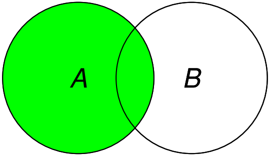

In A!

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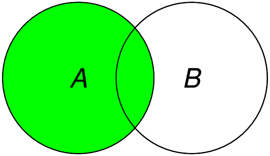

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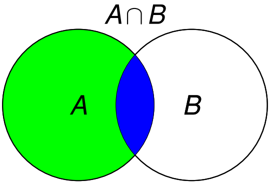

In A!

In B?

Must be in $A \cap B$.

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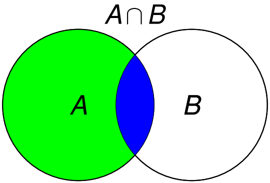

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In B?

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$$Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]}.$$

Conditional probability: example.

Two coin flips.

Conditional probability: example.

Two coin flips. First flip is heads.

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Two coin flips. First flip is heads. Probability of two heads?

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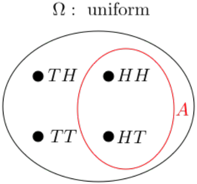

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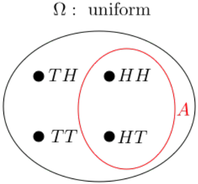

New sample space: A;

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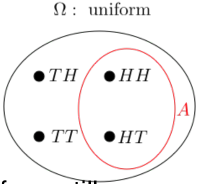

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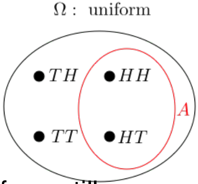

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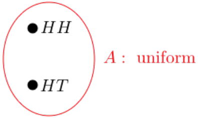

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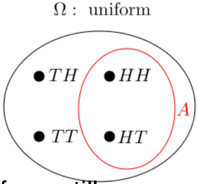

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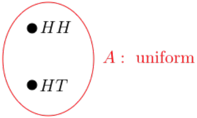

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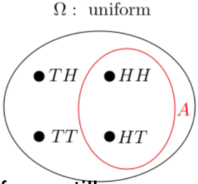

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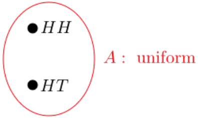

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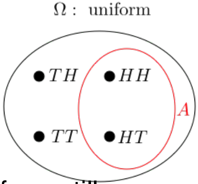

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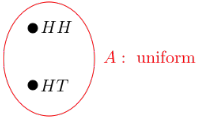

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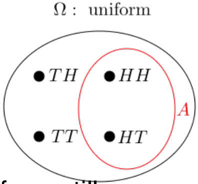

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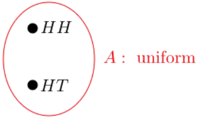

Event $B =$ two heads.

The probability of two heads if the first flip is heads.
The probability of B given A is $1/2$.

A similar example.

Two coin flips.

A similar example.

Two coin flips. At least one of the flips is heads.

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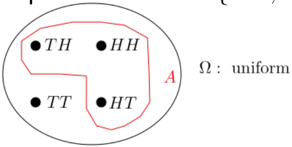

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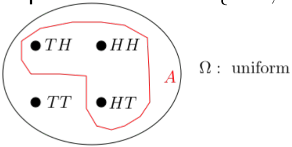

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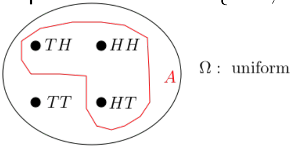

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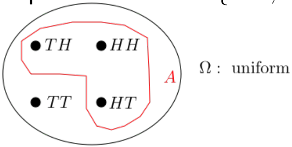

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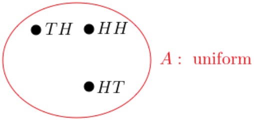

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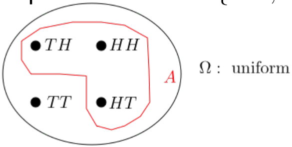

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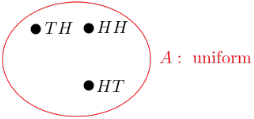

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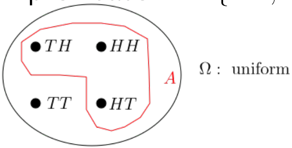

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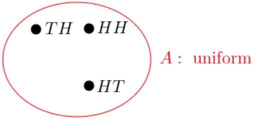

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The probability of two heads if at least one flip is heads.

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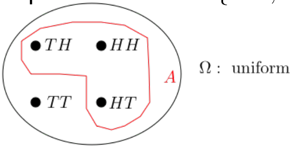

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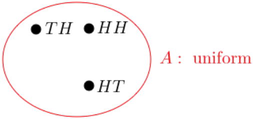

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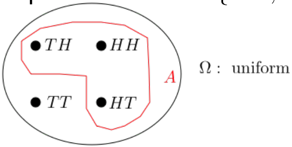

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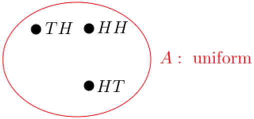

Event $B =$ two heads.

The probability of two heads if at least one flip is heads.
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Conditional Probability: A non-uniform example

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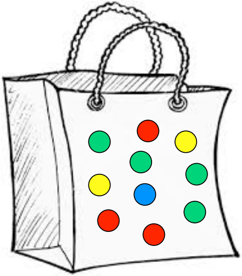
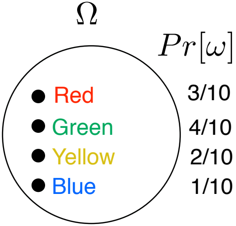


Physical experiment



Probability model

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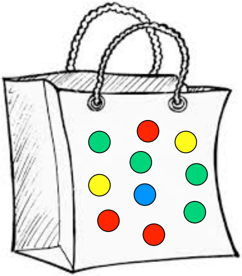

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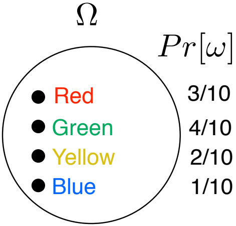

Probability model

$\Omega = \{$Red, Green, Yellow, Blue$\}$

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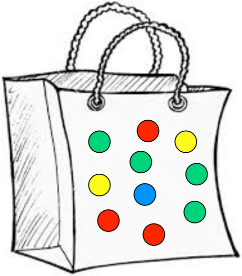

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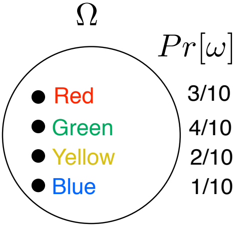

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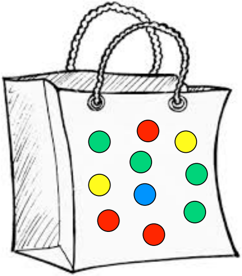
$$\Omega = \{\text{Red, Green, Yellow, Blue}\}$$

$$Pr[\text{Red}|\text{Red or Green}] =$$

Conditional Probability: A non-uniform example



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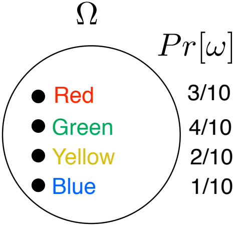

Probability model

$$\Omega = \{\text{Red, Green, Yellow, Blue}\}$$

$$Pr[\text{Red}|\text{Red or Green}] = \frac{Pr[\text{Red} \cap (\text{Red or Green})]}{Pr[\text{Red or Green}]}$$

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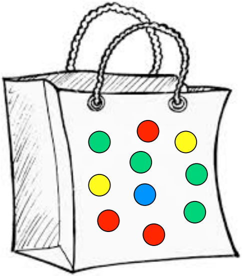

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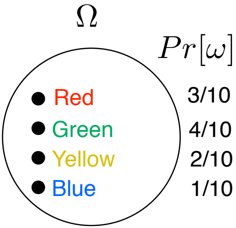

Probability model

$$\Omega = \{\text{Red, Green, Yellow, Blue}\}$$

$$Pr[\text{Red}|\text{Red or Green}] = \frac{Pr[\text{Red} \cap (\text{Red or Green})]}{Pr[\text{Red or Green}]} = \frac{3}{7}$$

$$Pr[\text{Blue}|\text{Red or Green}] =$$

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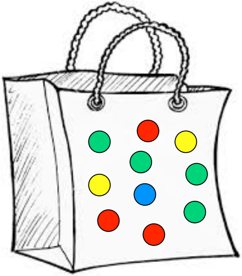

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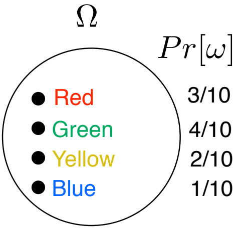

Probability model

$$\Omega = \{\text{Red, Green, Yellow, Blue}\}$$

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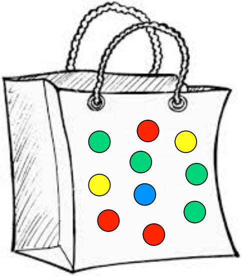
Conditional Probability: A non-uniform example



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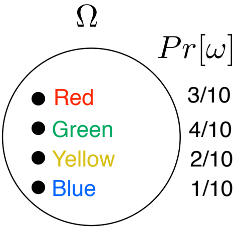

Probability model

$$\Omega = \{\text{Red, Green, Yellow, Blue}\}$$

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$$Pr[\text{Blue}|\text{Red or Green}] = \frac{Pr[\text{Blue} \cap (\text{Red or Green})]}{Pr[\text{Red or Green}]} = \frac{0}{7}$$

Another non-uniform example

Consider $\Omega = \{1, 2, \ldots, N\}$ with $Pr[n] = p_n$.

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Consider $\Omega = \{1, 2, \ldots, N\}$ with $Pr[n] = p_n$.
Let $A = \{3, 4\}, B = \{1, 2, 3\}$.

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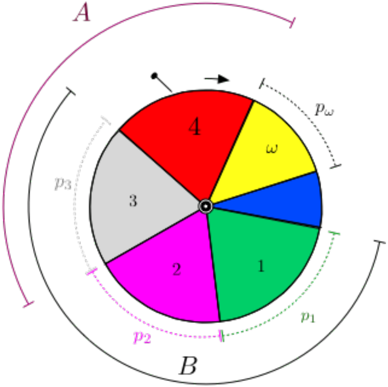

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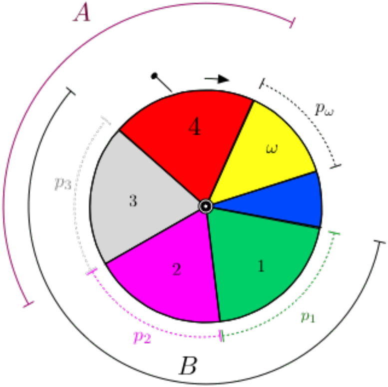

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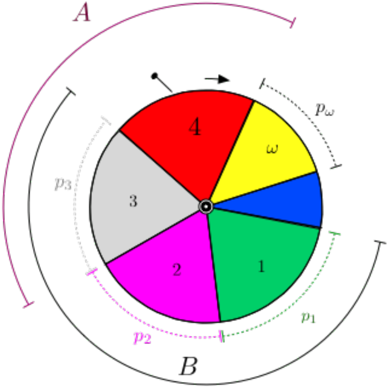

$$Pr[A|B] = \frac{p_3}{p_1 + p_2 + p_3} = \frac{Pr[A \cap B]}{Pr[B]}.$$

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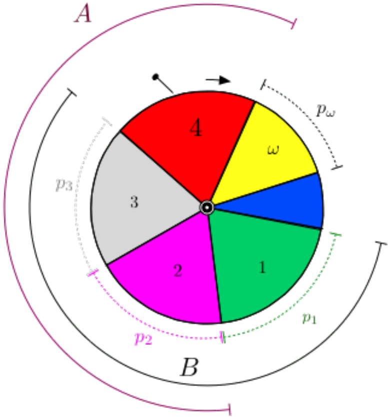

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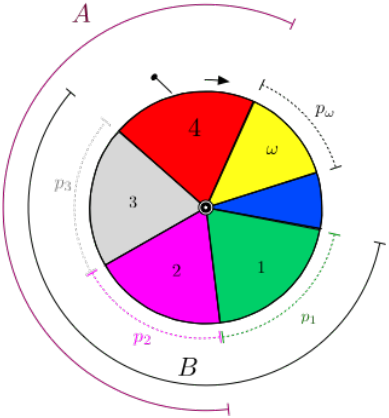

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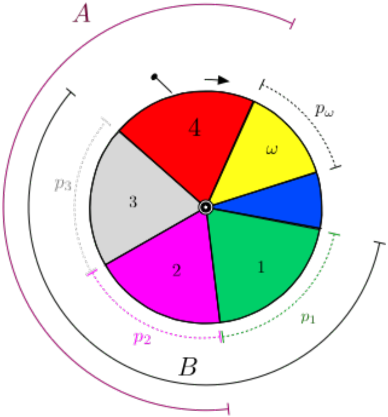

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Toss a red and a blue die, sum is 4,

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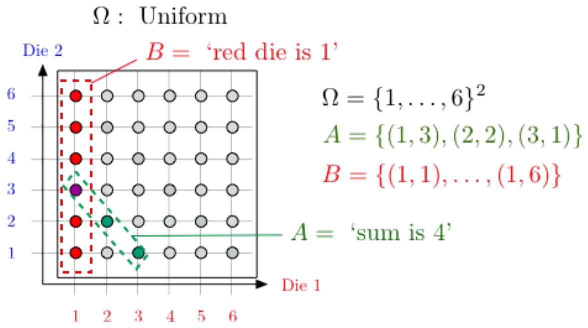

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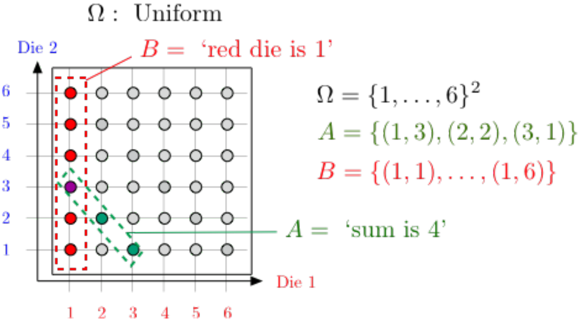

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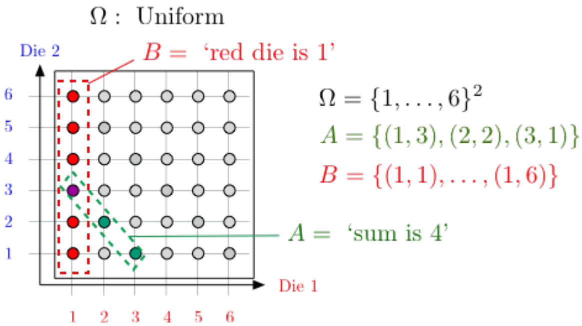

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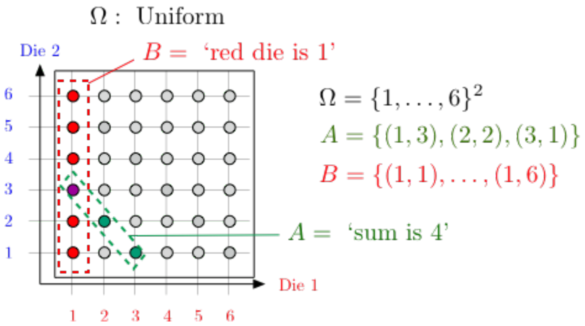

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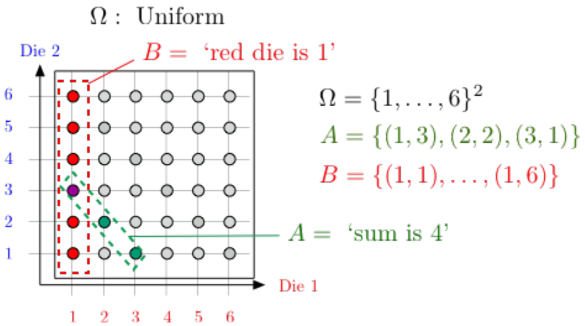

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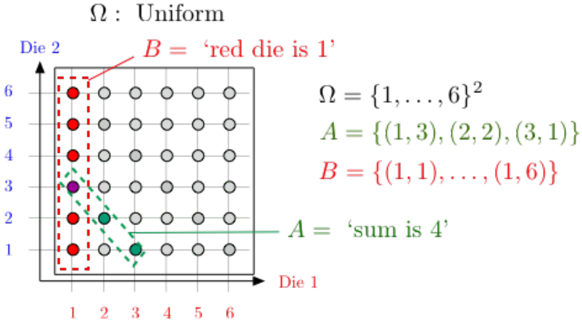

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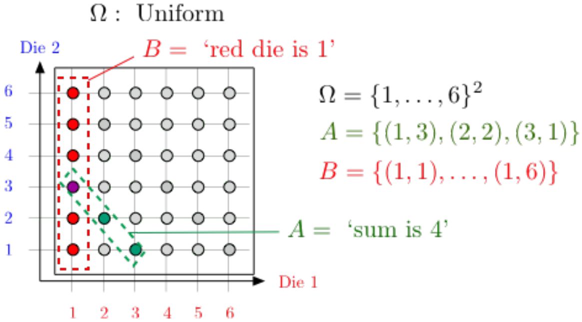

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B is more likely given *A*.

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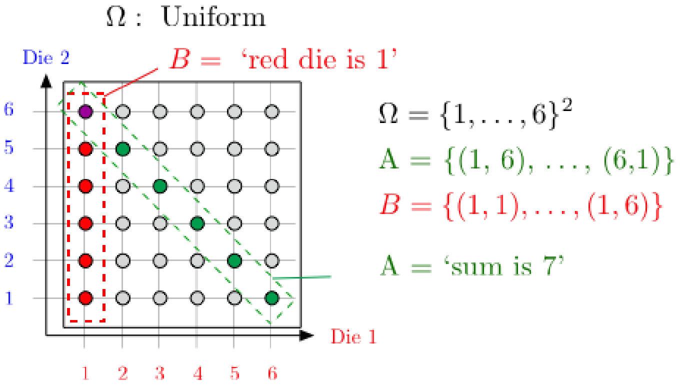

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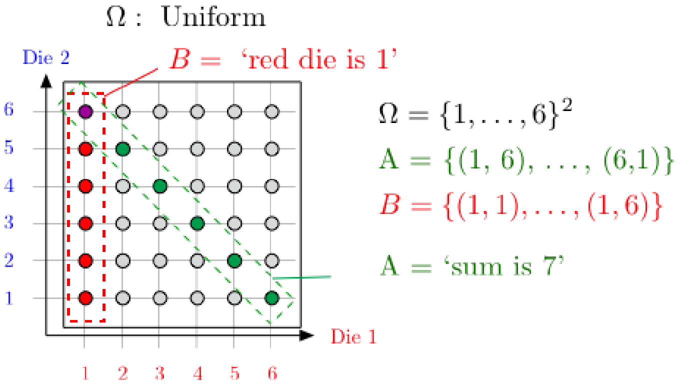

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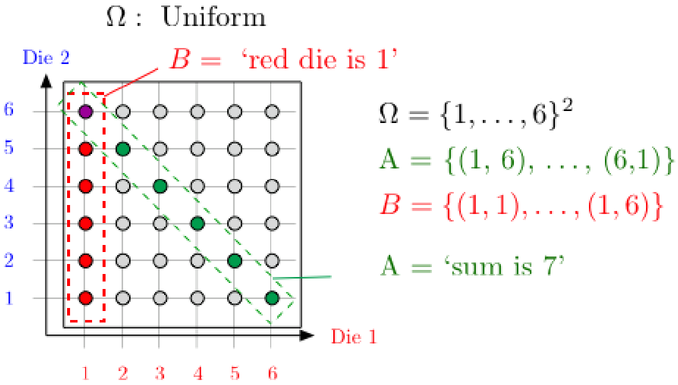

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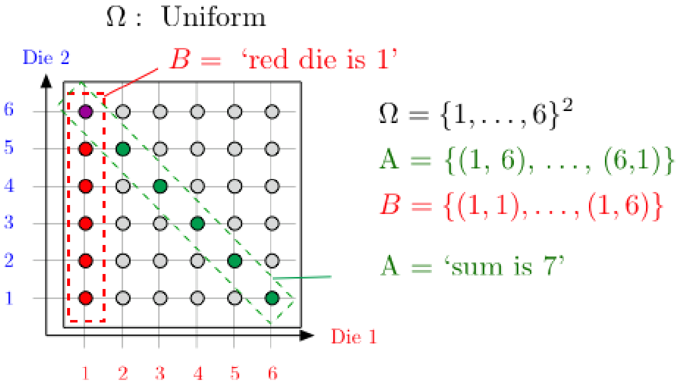

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Observing *A* does not change your mind about the likelihood of *B*.

Such empty: poll

Suppose I toss 3 balls into 3 bins.

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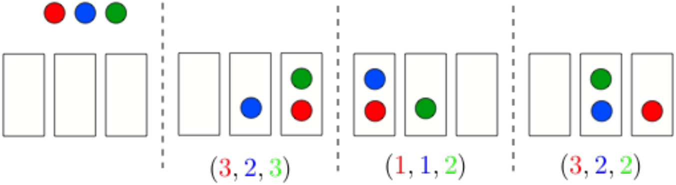

$\omega =$ (bin of red ball, bin of blue ball, bin of green ball)

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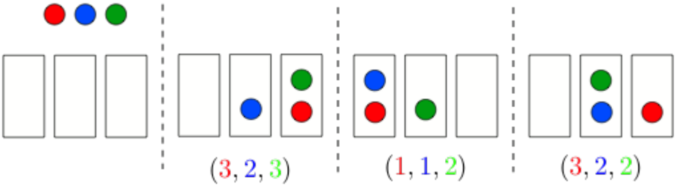

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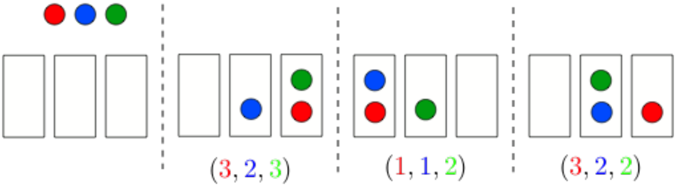

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What is $Pr[A|B]$?

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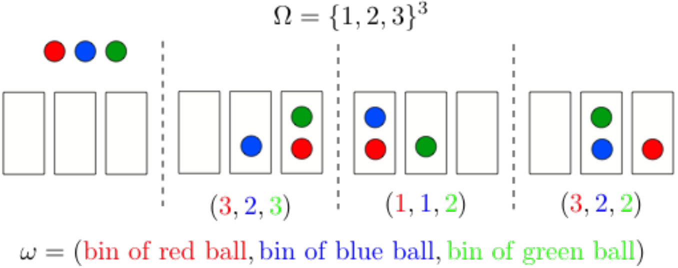

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(A) 1/27
(B) 8/27
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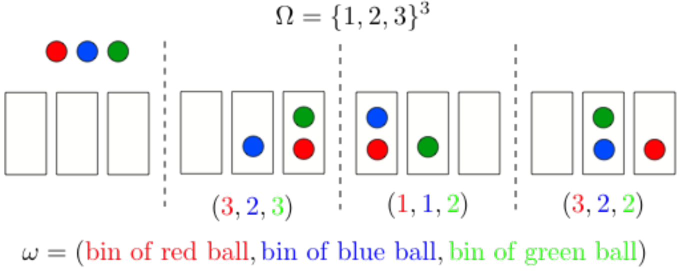

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Next slide.

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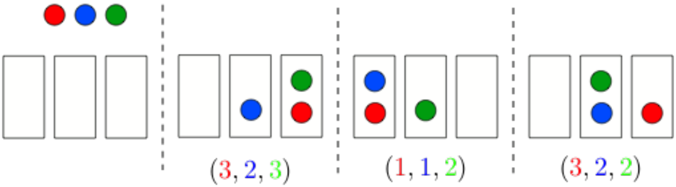

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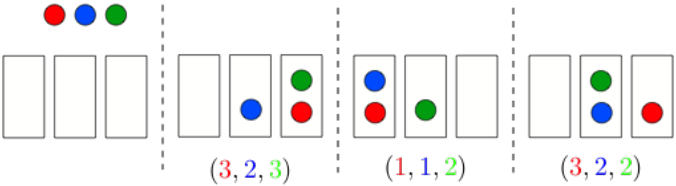

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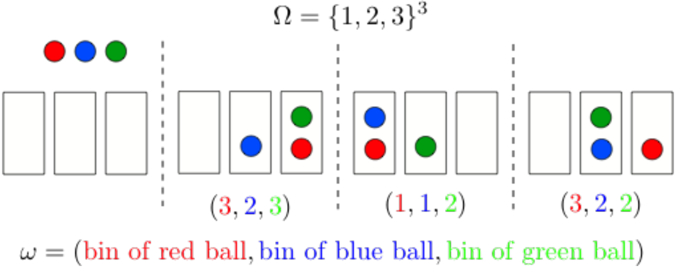

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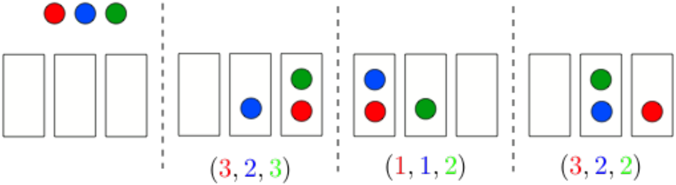

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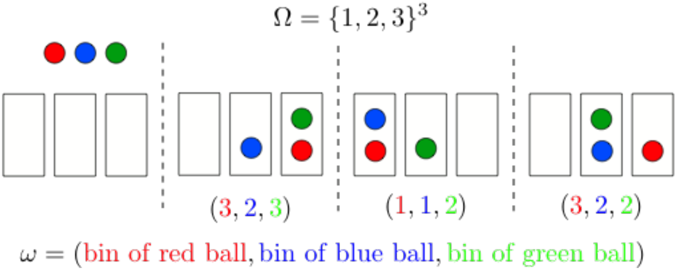

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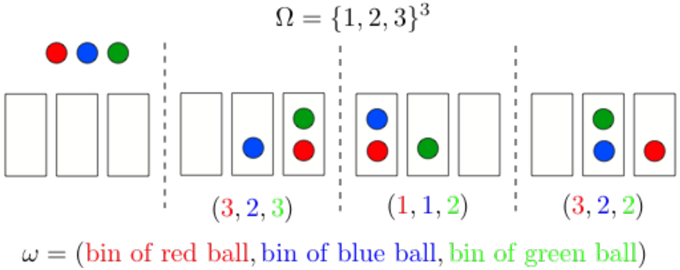

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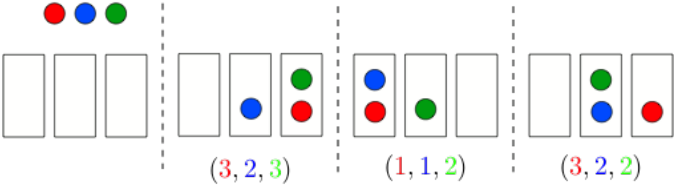

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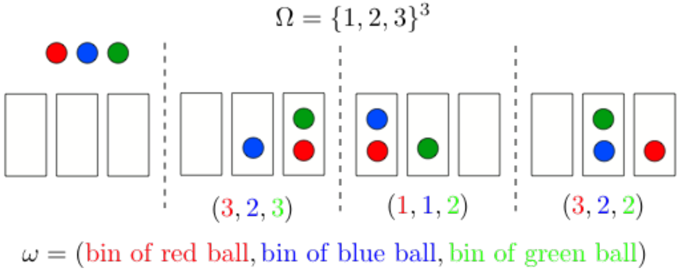

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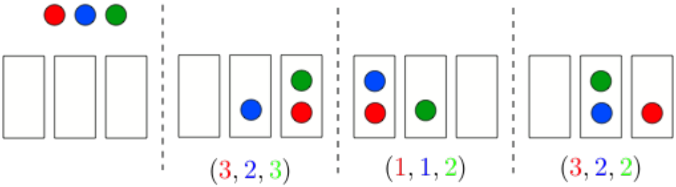

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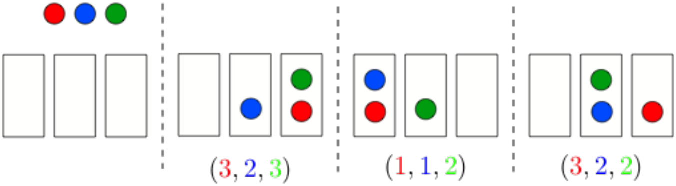

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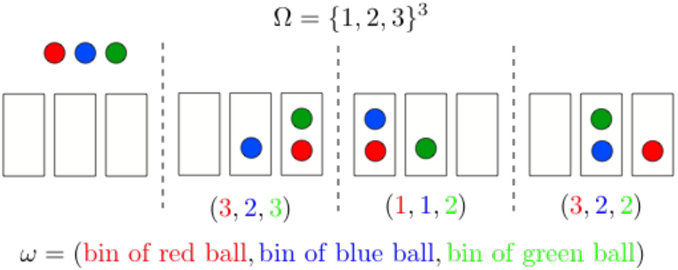

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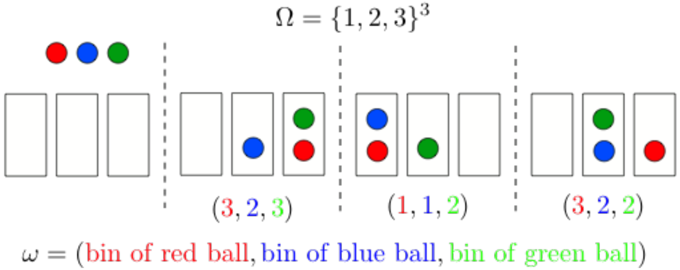

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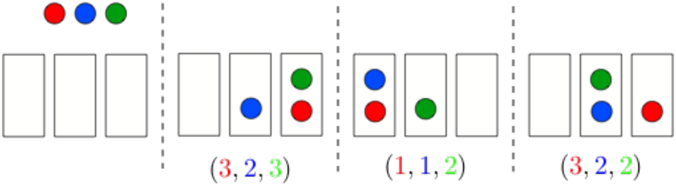

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A is less likely given B:

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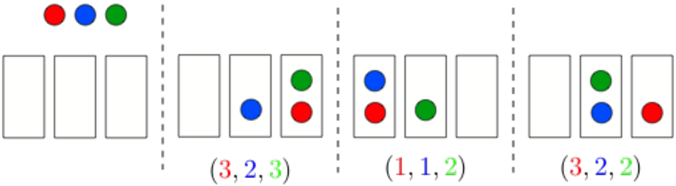

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A is less likely given B:
Second bin is empty $\implies$ first is more likely to have ball(s).

Gambler's fallacy.

Flip a fair coin 51 times.

Gambler’s fallacy.

Flip a fair coin 51 times.
A = “first 50 flips are heads”

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$A = \{HH\cdots HT, HH\cdots HH\}$

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$Pr[B|A]$?

$A = \{HH \cdots HT, HH \cdots HH\}$
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Uniform probability space.

Gambler's fallacy.

Flip a fair coin 51 times.
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The likelihood of 51st heads does not depend on the previous flips.

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so that the result holds for $n+1$. $\square$

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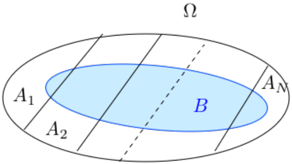
More about such questions later. For fun, check "N. Taleb: Fooled by randomness."

Total probability with Conditional Probability.

Assume that Ω is the union of the disjoint sets $A_1, \ldots, A_N$.



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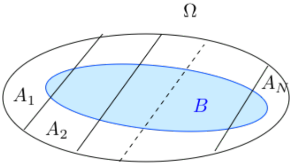

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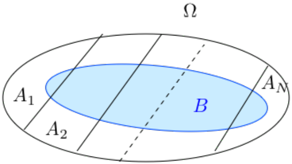

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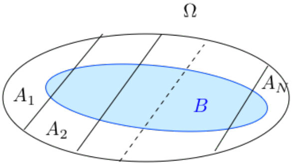

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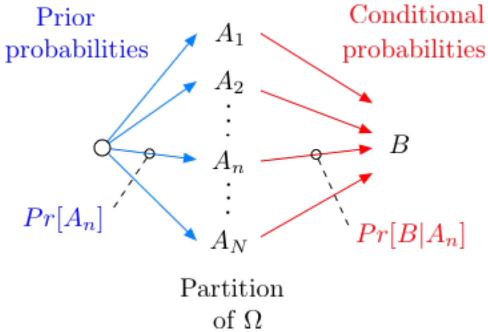

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You flip your coin and it yields heads.

What is the probability that it is fair?

Analysis:

$$A = \text{ 'coin is fair'}, B = \text{ 'outcome is heads'}$$

We want to calculate $P[A|B]$.

We know $P[B|A] = 1/2, P[B|\bar{A}] = 0.6, Pr[A] = 1/2 = Pr[\bar{A}]$

Now,

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Thus,

$$Pr[A|B] = \frac{Pr[A]Pr[B|A]}{Pr[B]} = \frac{(1/2)(1/2)}{(1/2)(1/2) + (1/2)0.6} \approx 0.45.$$

Independence:poll

Definition: Two events A and B are **independent** if

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Which Examples are independent?

(A) Roll two dice, $A =$ sum is 7 and $B =$ red die is 1.
(B) Roll two dice, $A =$ sum is 3 and $B =$ red die is 1.
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Conditional Probability: Pictures

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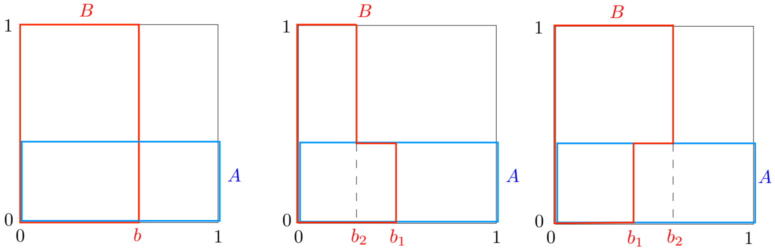
Illustrations: Pick a point uniformly in the unit square



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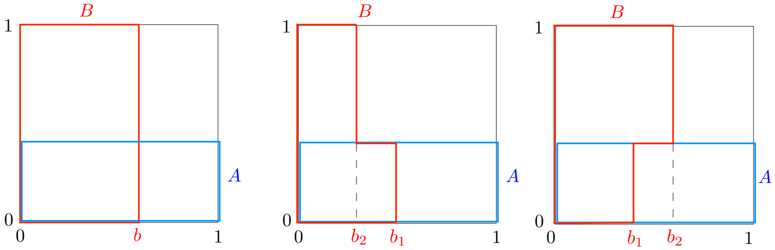

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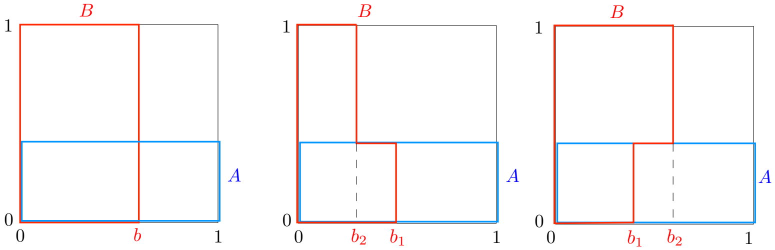

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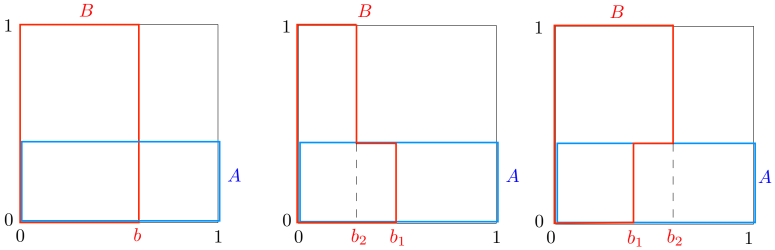

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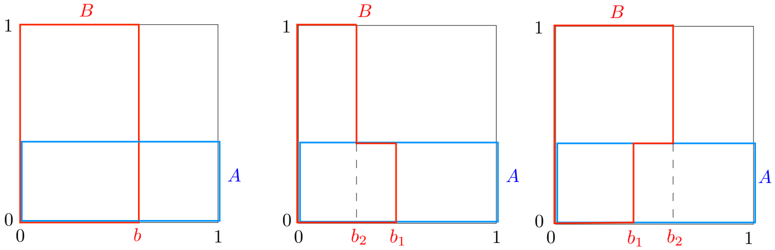

- Left: A and B are independent. $Pr[B] = b$;

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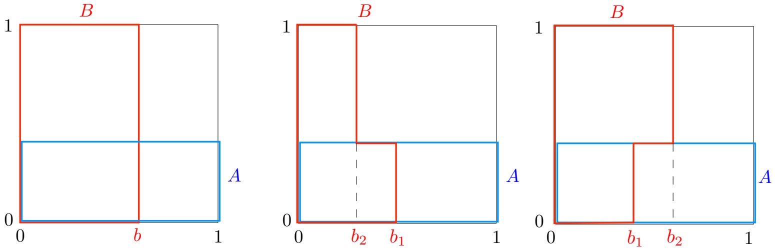

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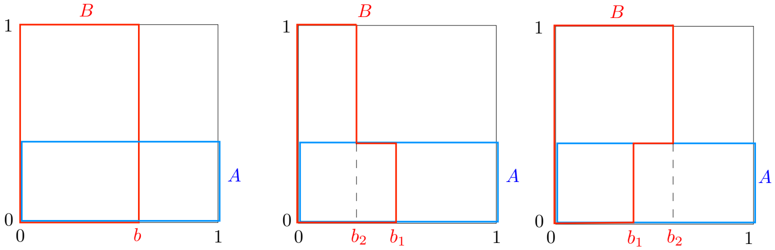

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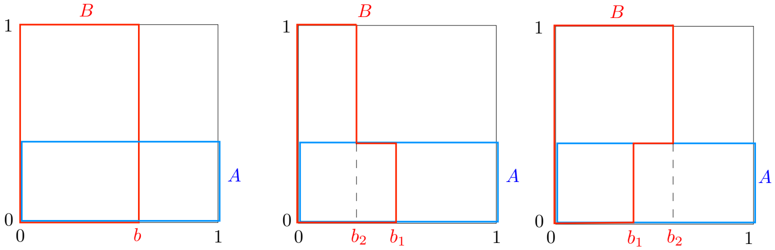

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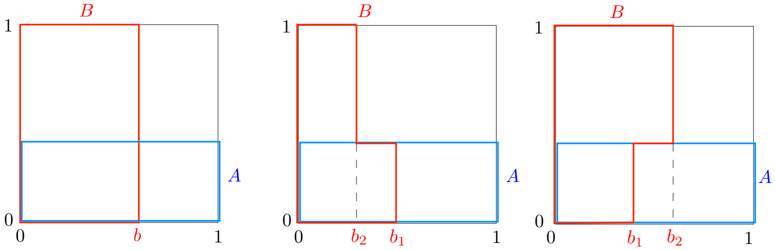

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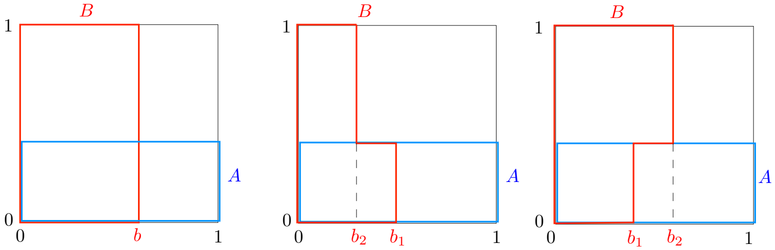

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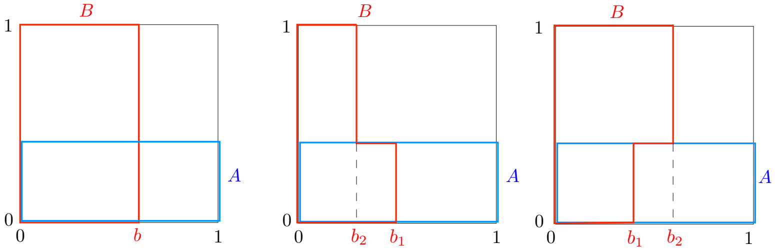

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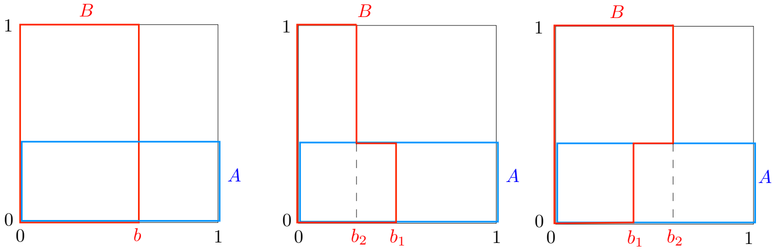

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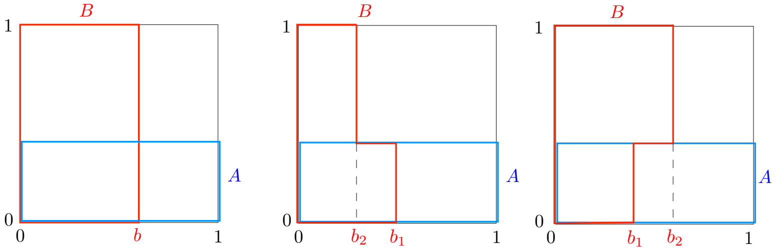

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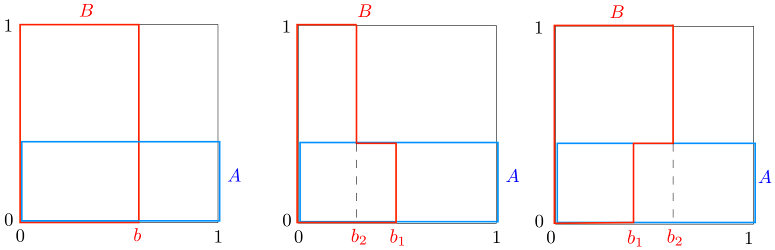

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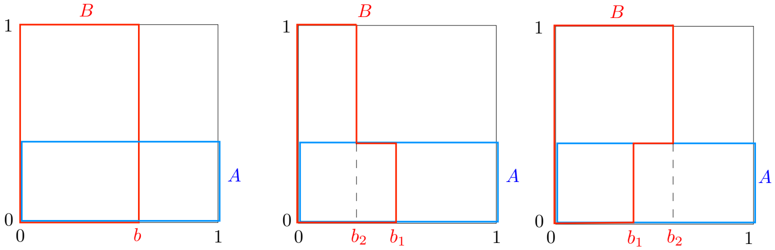

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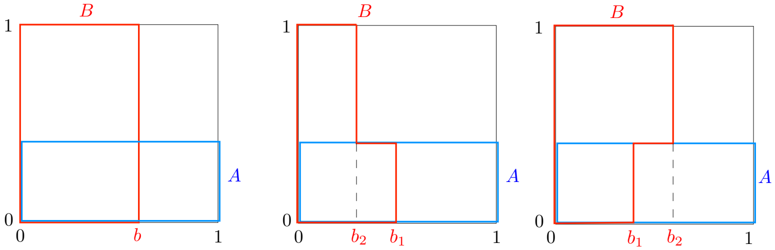

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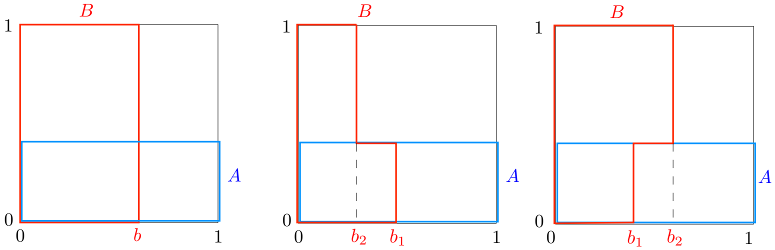

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