Today

Probability:

Keep building it formally..

And our intuition.

Poll: blows my mind.

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Also, "cuz" $==$ "because"

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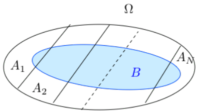

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(A), (B), and (C)

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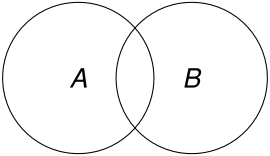
Definition: The **conditional probability** of B given A is

$$Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]}$$



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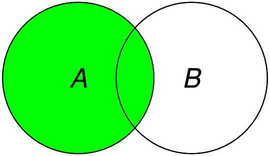

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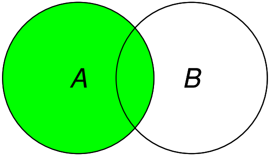

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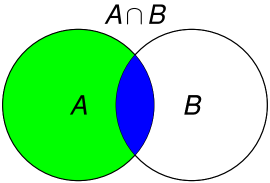

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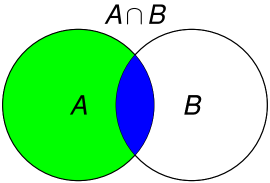

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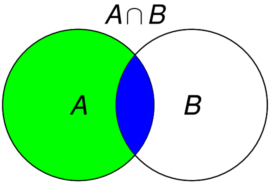

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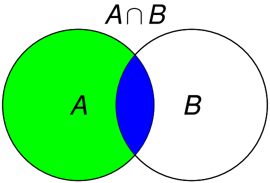

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Thus,

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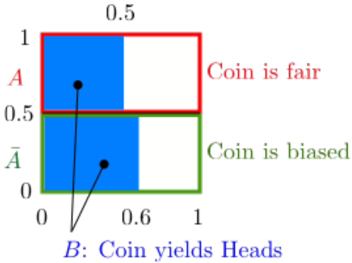

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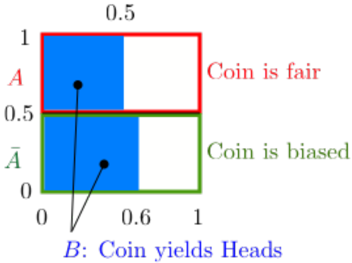

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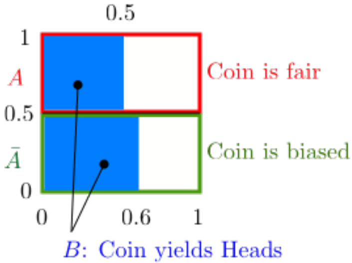

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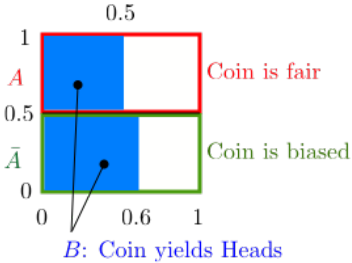

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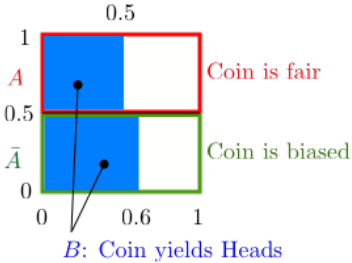

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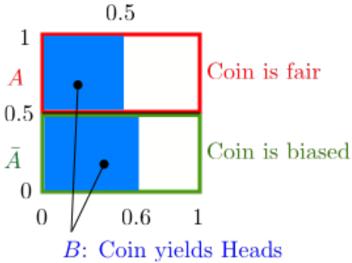

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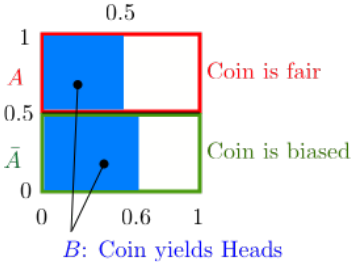

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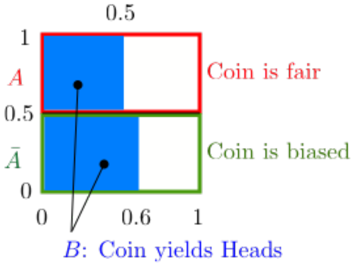

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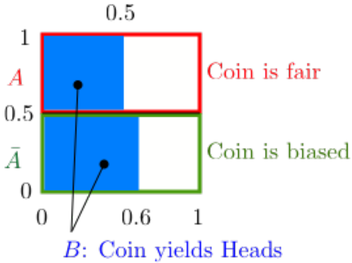

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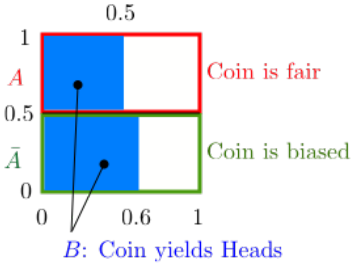

Pick a point uniformly at random in the unit square. Then

$$Pr[A] = 0.5; Pr[\bar{A}] = 0.5$$
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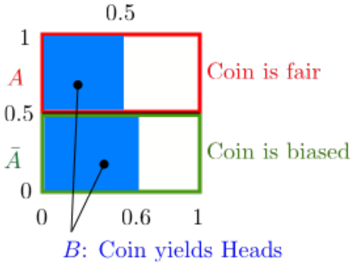

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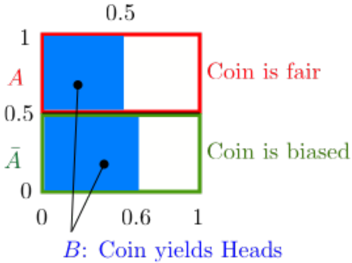

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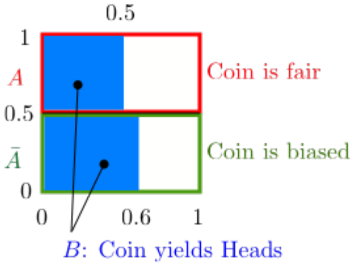

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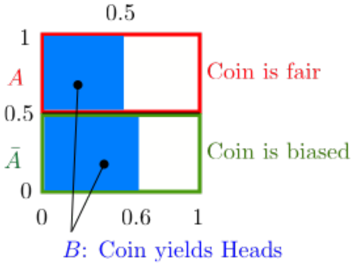
Bayes and Biased Coin



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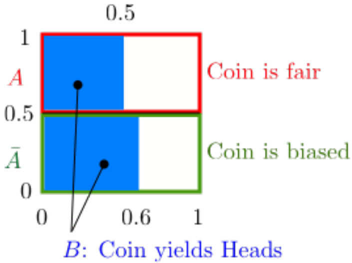
Bayes and Biased Coin



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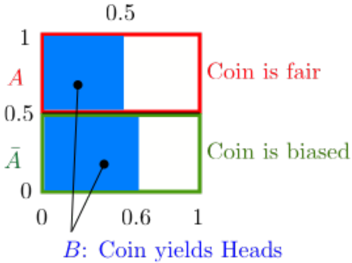
Bayes and Biased Coin



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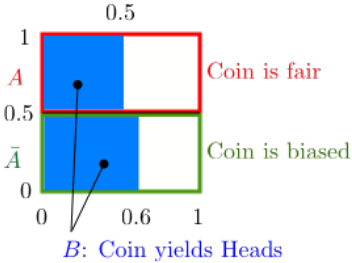
Bayes and Biased Coin



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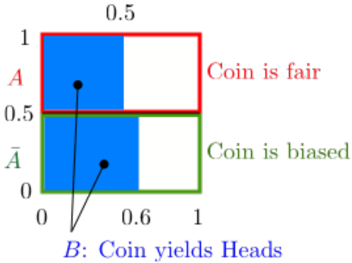

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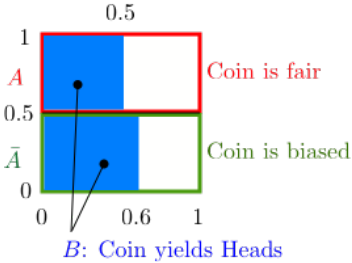
$$\approx 0.46$$

Bayes and Biased Coin



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$$Pr[A|B] = \frac{0.5 \times 0.5}{0.5 \times 0.5 + 0.5 \times 0.6} = \frac{Pr[A]Pr[B|A]}{Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}]}$$
$$\approx 0.46 = \text{fraction of } B \text{ that is inside } A$$

Bayes: General Case

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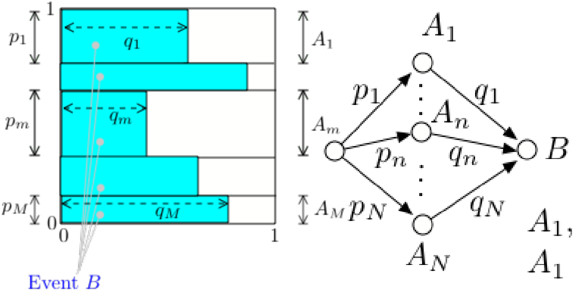

$$p_n = Pr[A_n]$$
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$A_1, \ldots, A_N$ disjoint

$$A_1 \cup \cdots \cup A_N = \Omega$$

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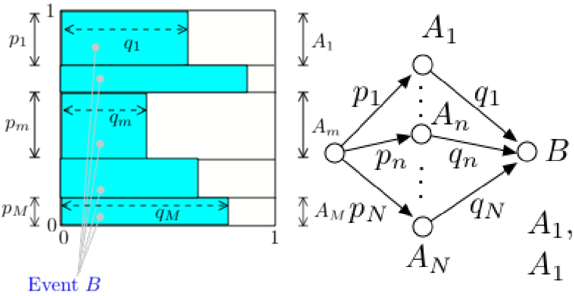

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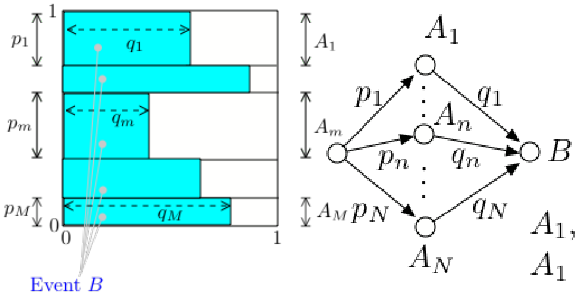

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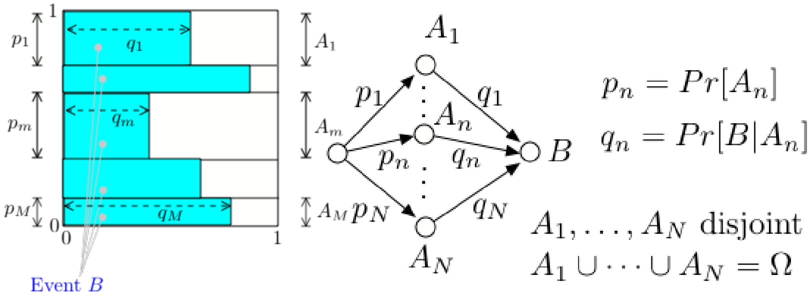

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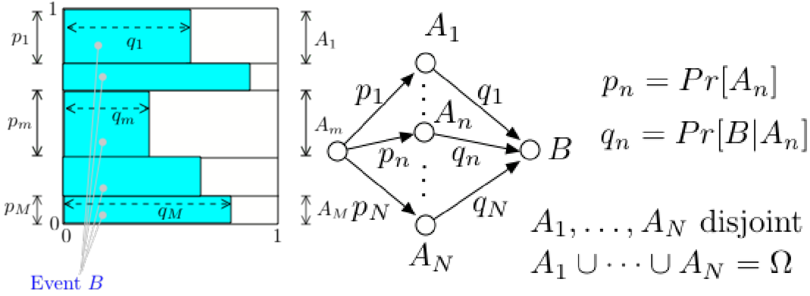

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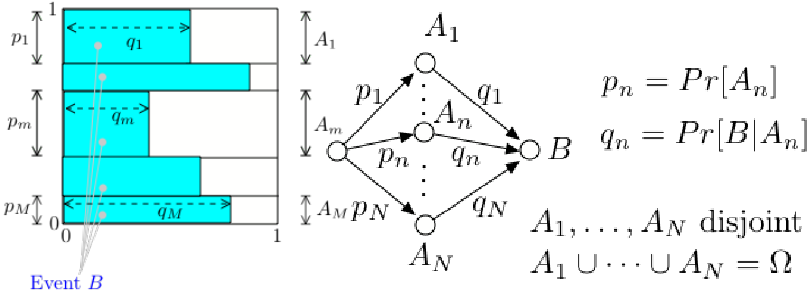

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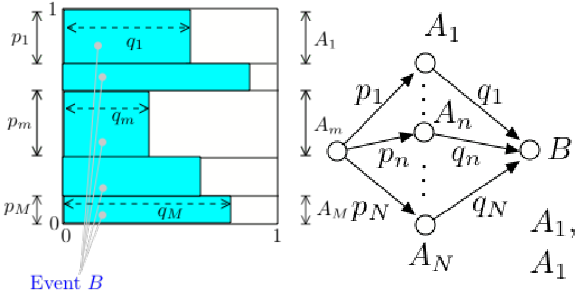

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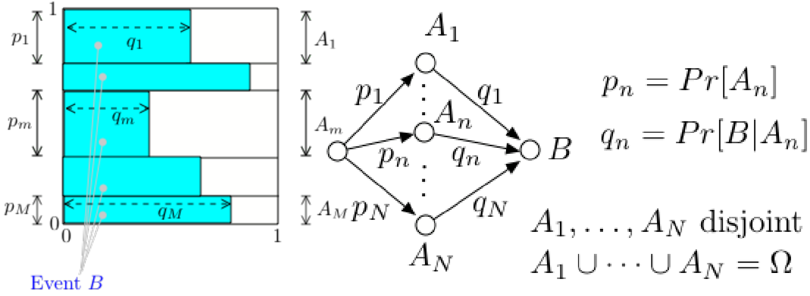

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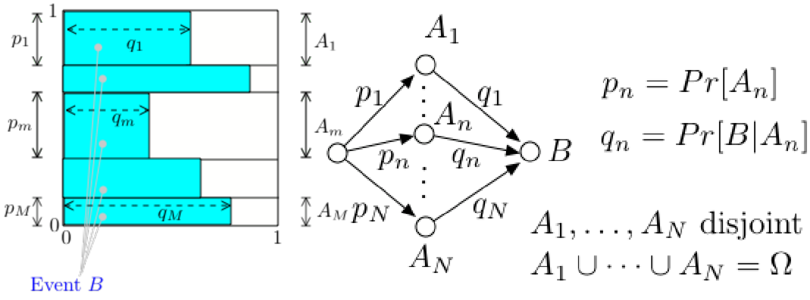
Bayes: General Case



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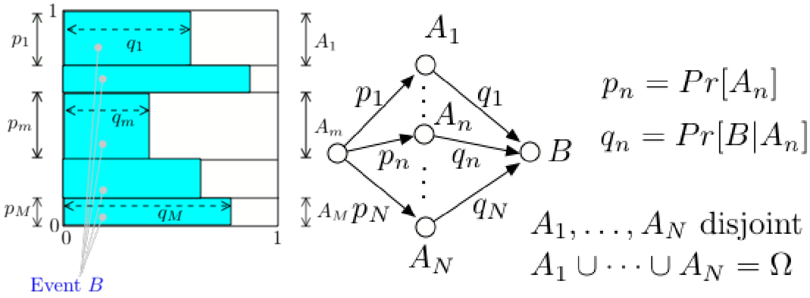
Bayes: General Case



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Bayes Rule

A general picture: We imagine that there are N possible causes $A_1, \ldots, A_N$.

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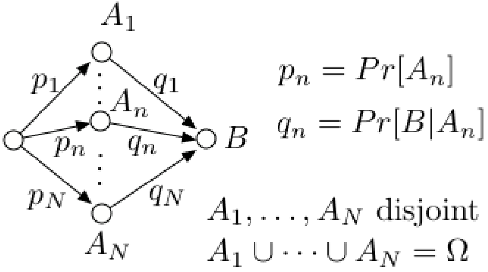

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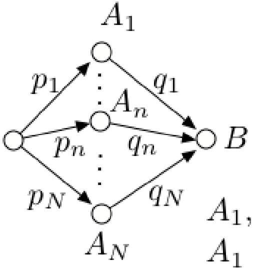

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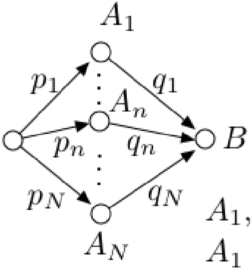

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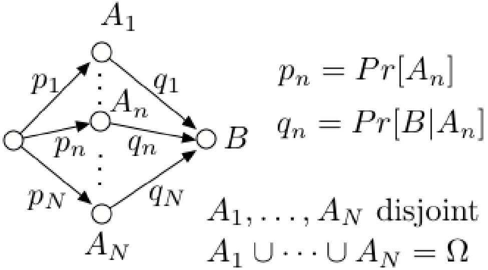

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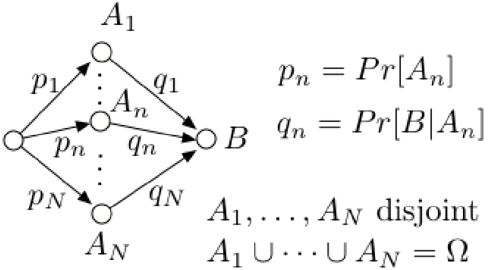

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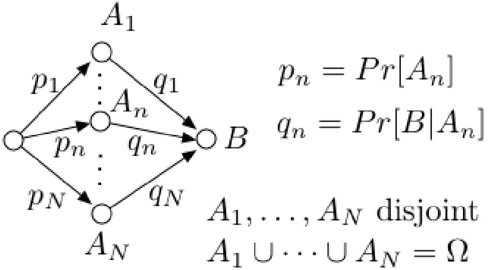

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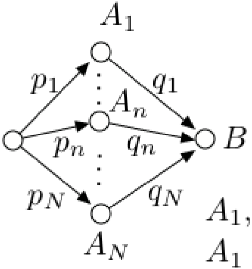

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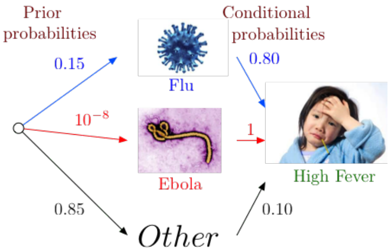
$$Pr[A_n|B] = \frac{Pr[B|A_n] Pr[A_n]}{Pr[B]}.$$

Why do you have a fever?



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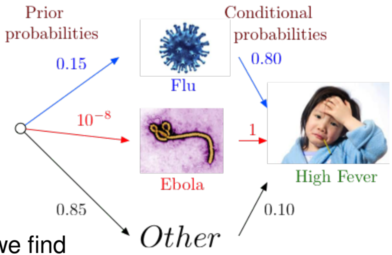

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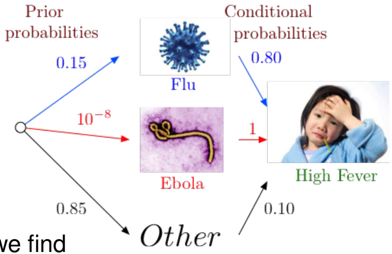

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$$Pr[\text{Flu}|\text{High Fever}] = \frac{0.15 \times 0.80}{0.15 \times 0.80 + 10^{-8} \times 1 + 0.85 \times 0.1} \approx 0.58$$

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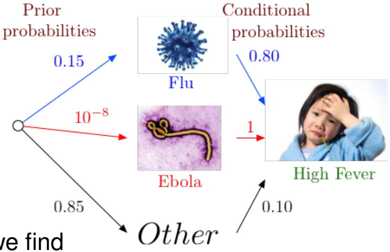

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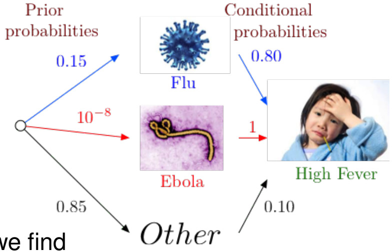

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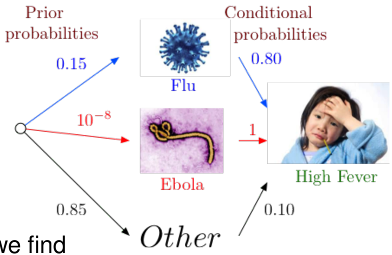
$$Pr[\text{Other}|\text{High Fever}] = \frac{0.85 \times 0.1}{0.15 \times 0.80 + 10^{-8} \times 1 + 0.85 \times 0.1} \approx 0.42$$

Why do you have a fever?



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The values $0.58, 5 \times 10^{-8}, 0.42$ are the posterior probabilities.

Why do you have a fever?

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Our “Bayes’ Square” picture:

Why do you have a fever?

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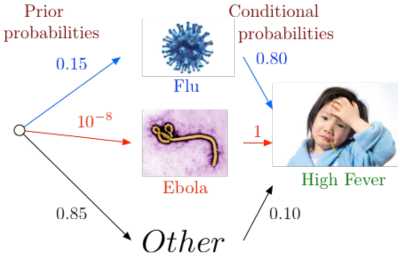
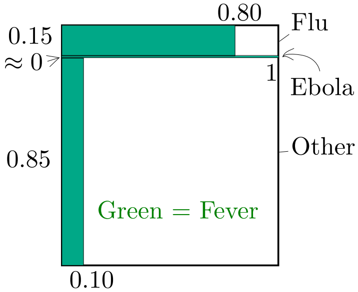




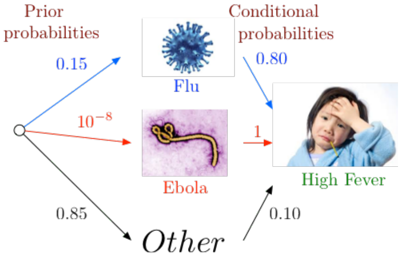
58% of Fever = Flu

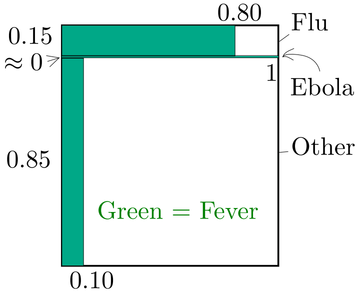
≈ 0% of Fever = Ebola

42% of Fever = Other

Why do you have a fever?

Our “Bayes’ Square” picture:





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≈ 0% of Fever = Ebola

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Note that even though $Pr[\text{Fever}|\text{Ebola}] = 1$,

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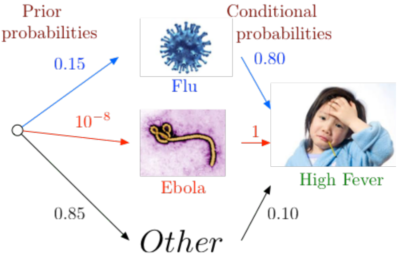
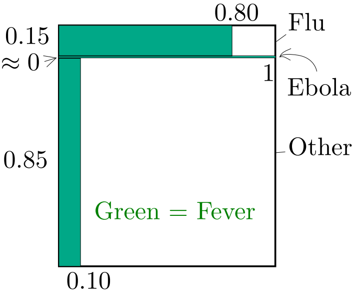
Our “Bayes’ Square” picture:





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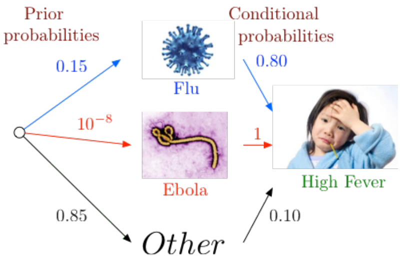

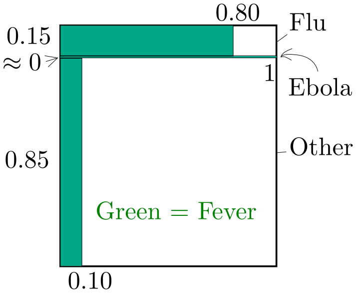

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This example shows the importance of the prior probabilities.

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Bayes' Rule Operations

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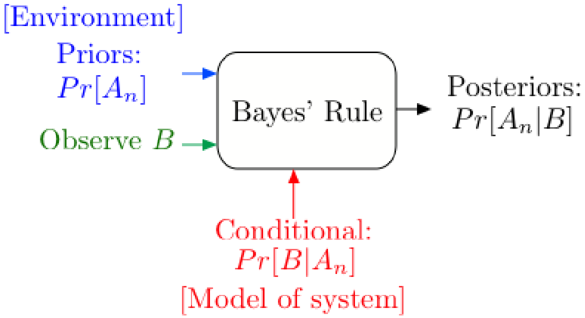

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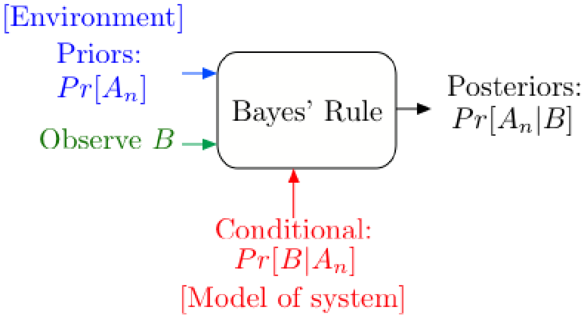

Bayes' Rule: canonical example of how information changes our opinions.

Thomas Bayes

Thomas Bayes

Portrait used of Bayes in a 1936 book,[1] but it is doubtful whether the portrait is actually of him.[2] No earlier portrait or claimed portrait survives.

Born	c. 1701 London, England
Died	7 April 1761 (aged 59) Tunbridge Wells, Kent, England
Residence	Tunbridge Wells, Kent, England
Nationality	English
Known for	Bayes' theorem

Source: Wikipedia.

Thomas Bayes

A

FIG. 3. *Joshua Bayes (1671–1746).*

$Pr[B|A]$ →

Max over B

B

Thomas Bayes?

A Bayesian picture of Thomas Bayes.

Testing for disease.

Random Experiment: Pick a random male.

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$$Pr[A|B]???$$

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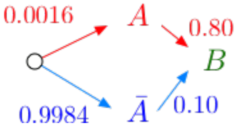

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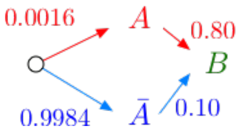

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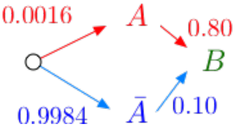

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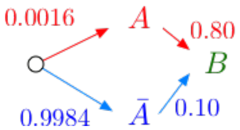

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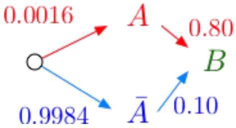

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A 1.3% chance of prostate cancer with a positive PSA test.

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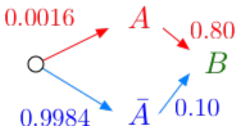

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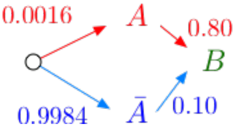

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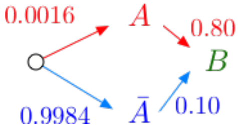

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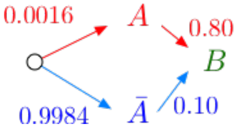

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Death.

Conditional Probability: Pictures/Poll.

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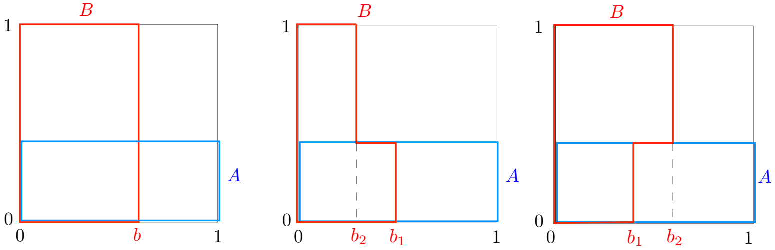
Illustrations: Pick a point uniformly in the unit square



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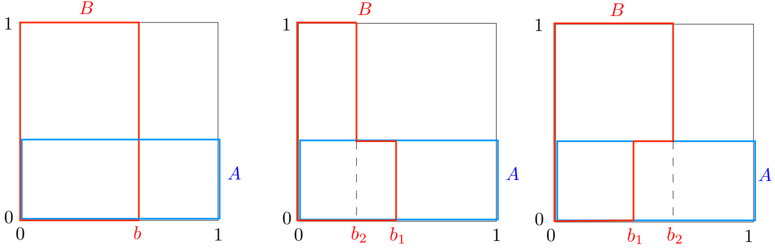

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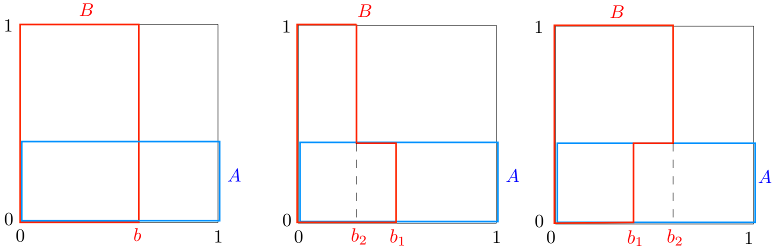

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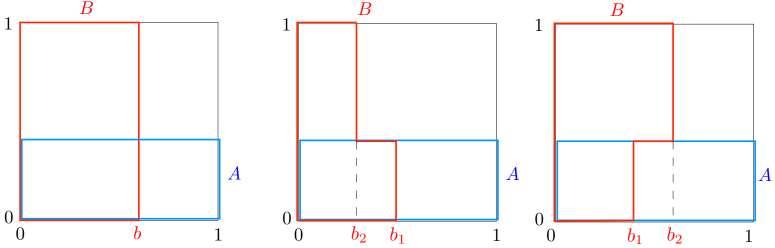

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See next slide.

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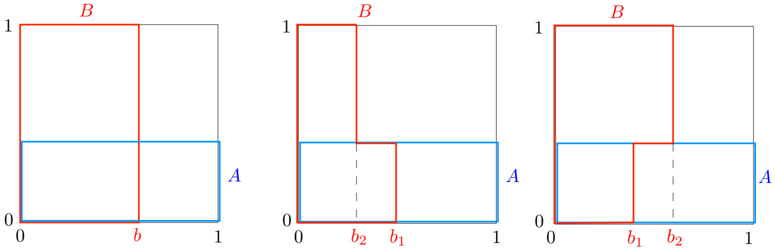

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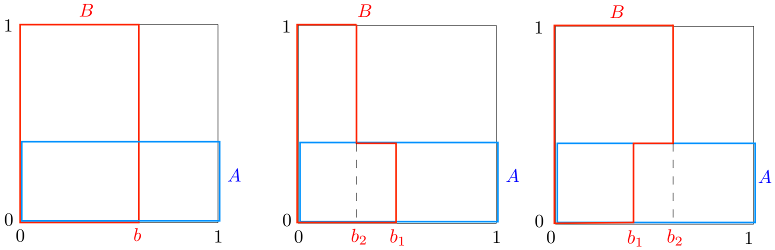

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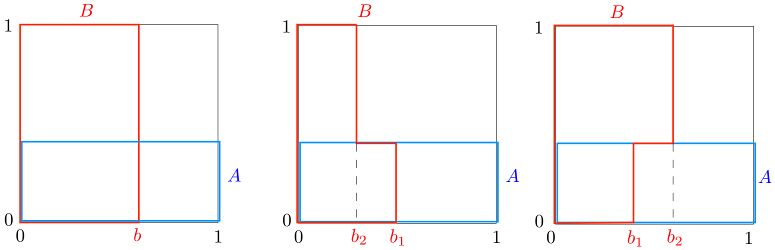

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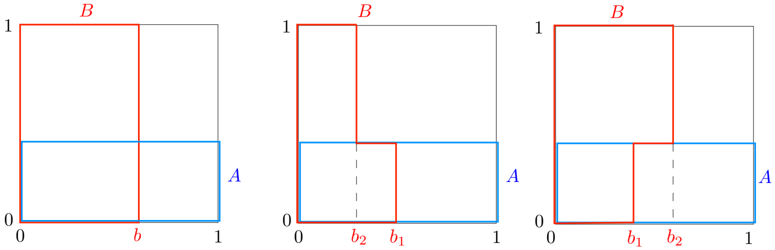

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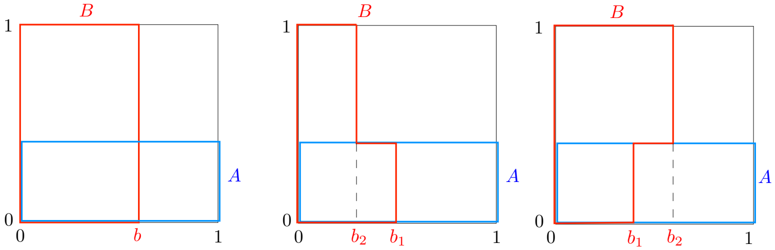

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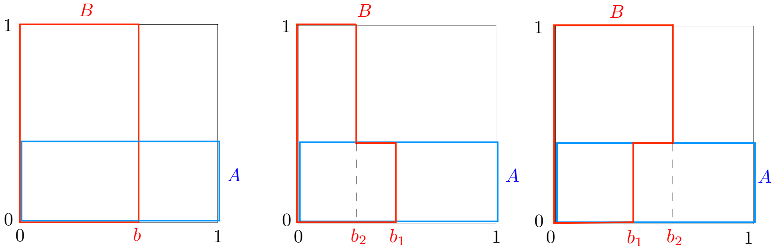

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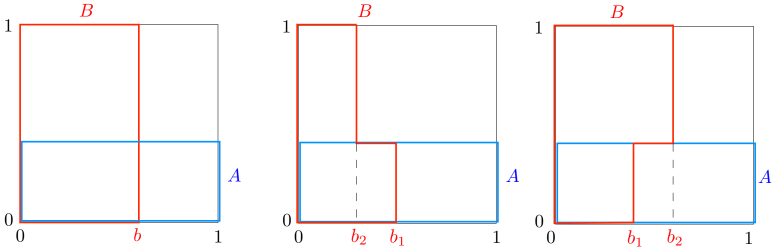

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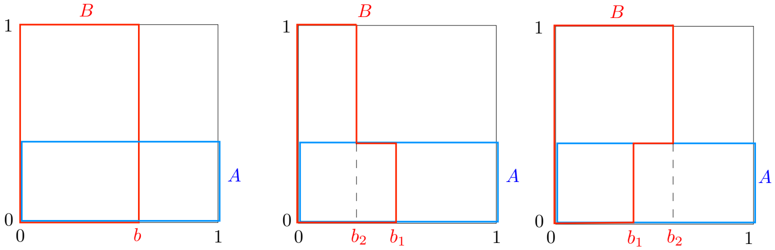

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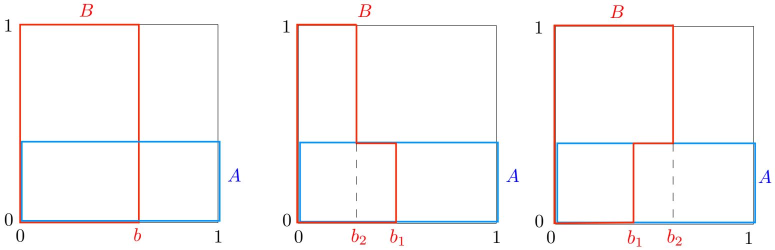

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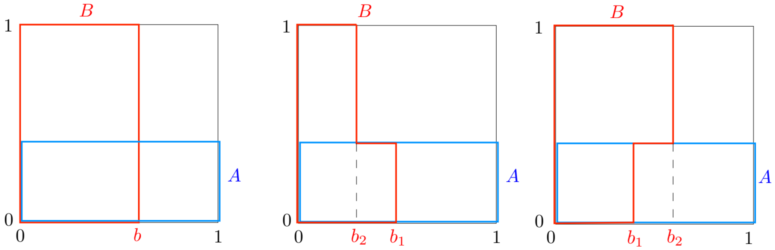

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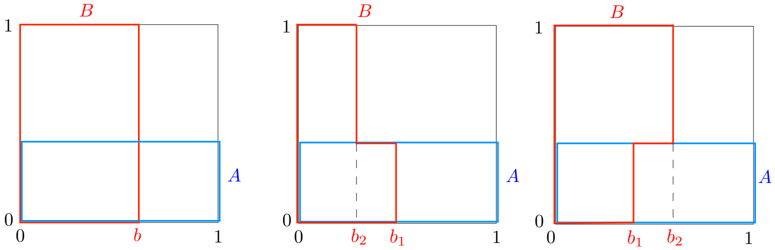

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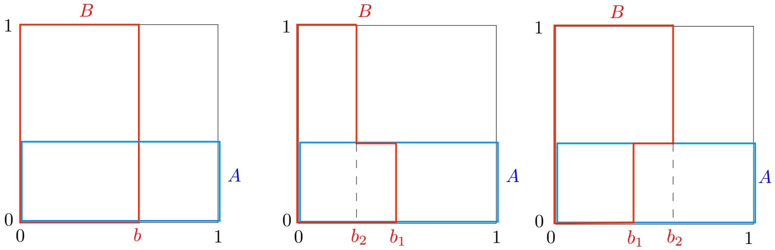

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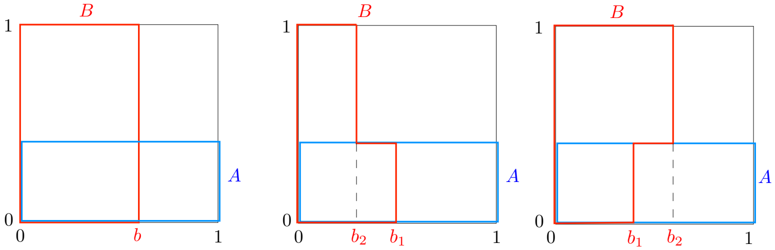

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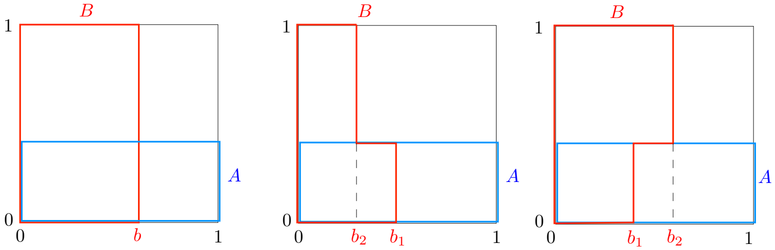

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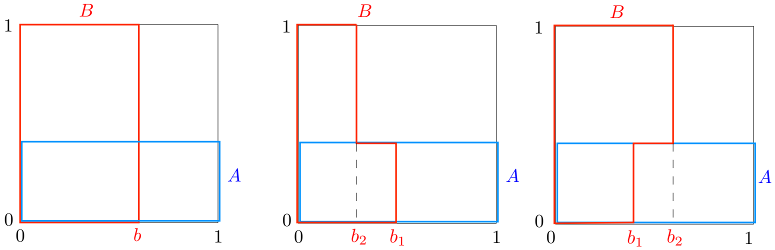

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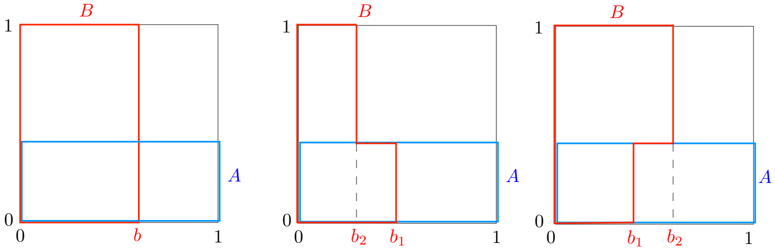

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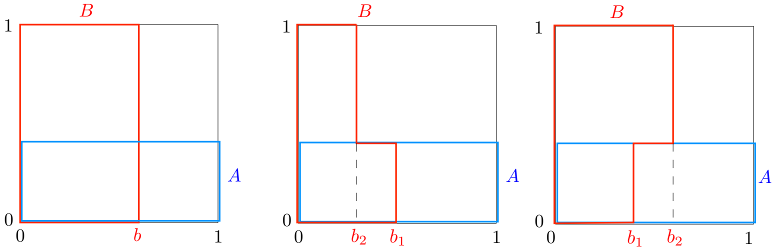

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Quick Review

Events, Conditional Probability, Independence, Bayes' Rule

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Events, Conditional Probability, Independence, Bayes' Rule

Key Ideas:

- Conditional Probability:

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- All these are possible:

  $Pr[A|B] < Pr[A]; Pr[A|B] > Pr[A]; Pr[A|B] = Pr[A]$.

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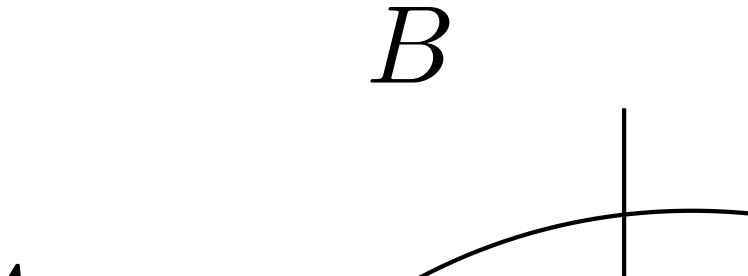

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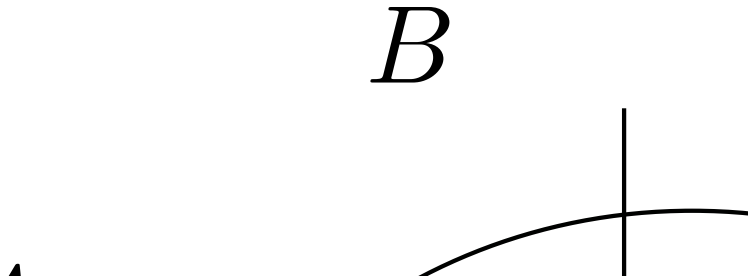

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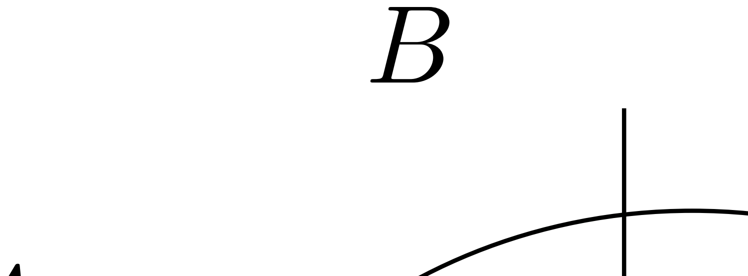

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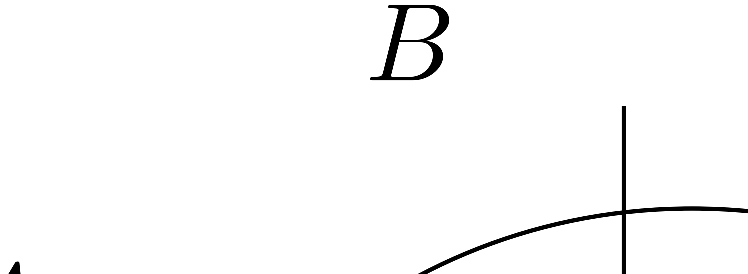

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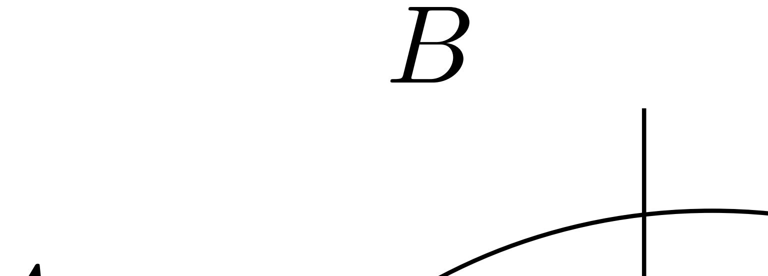

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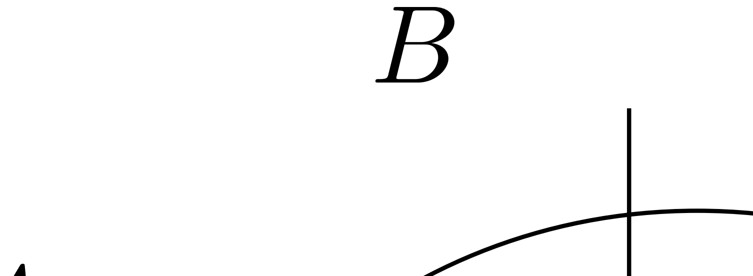

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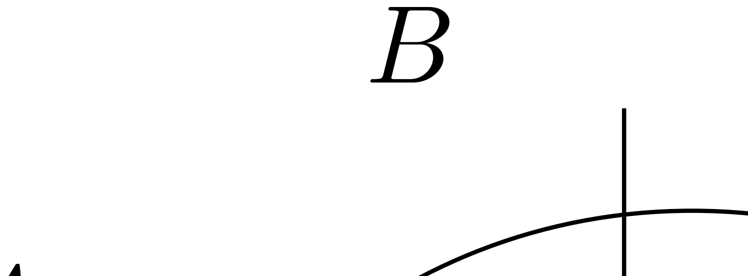

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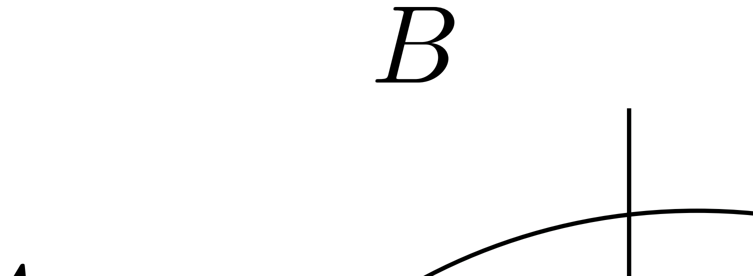

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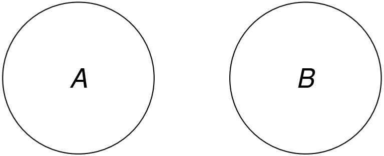

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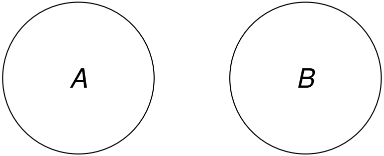

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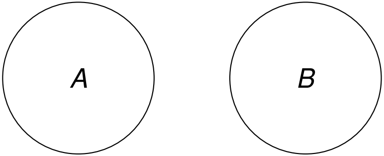

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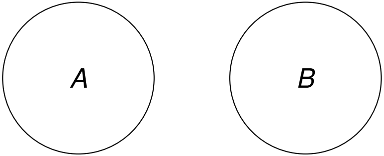

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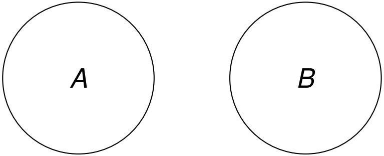

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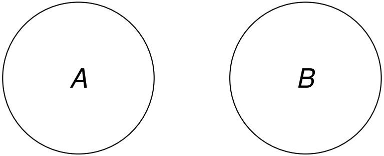

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Independent? $Pr[A] \neq Pr[A|B]$.

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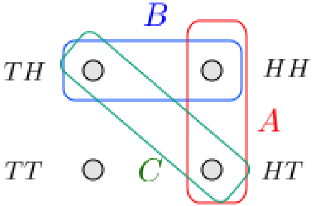

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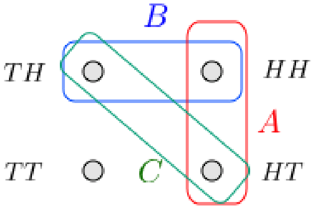

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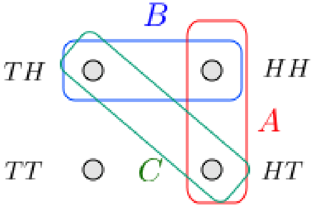

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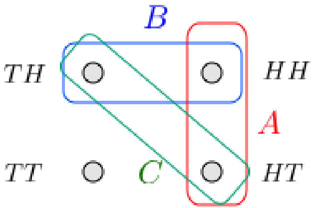

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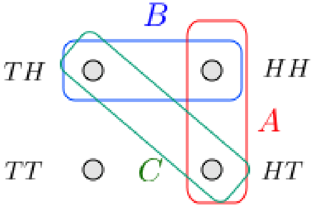

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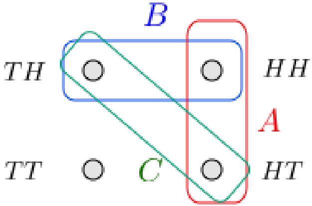

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False: If A did not say anything about C and B did not say anything about C, then $A \cap B$ would not say anything about C.

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Example: Flip a fair coin forever. Let A_n = 'coin n is H.' Then the events A_n are mutually independent.

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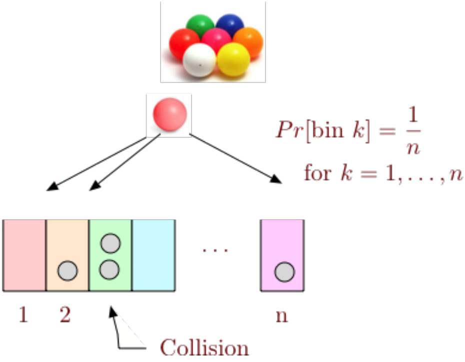

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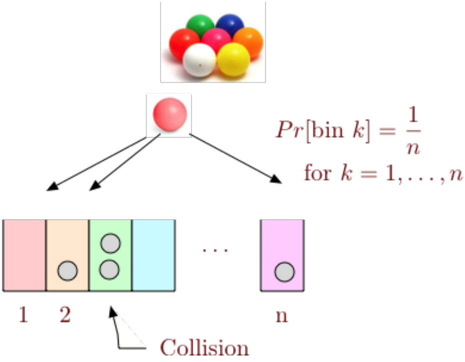

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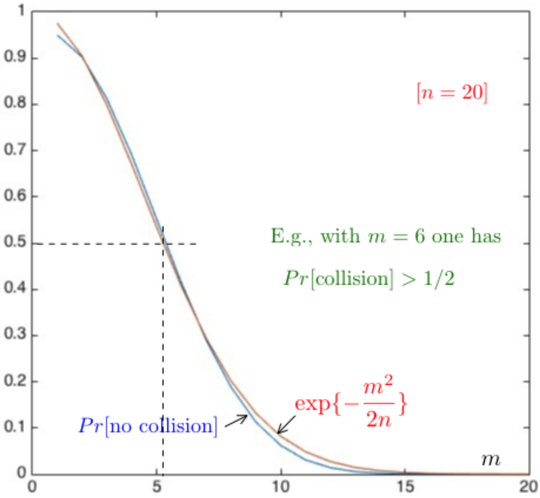

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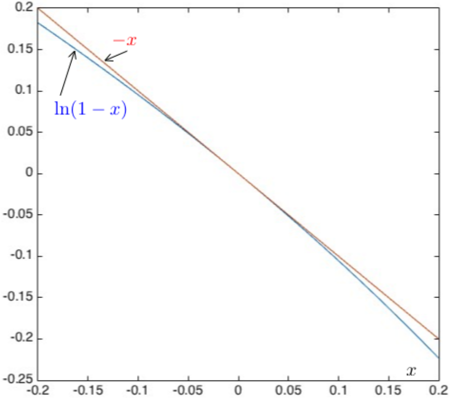
$^{(\dagger)}$ $1 + 2 + \cdots + m - 1 = (m-1)m/2$.

Approximation



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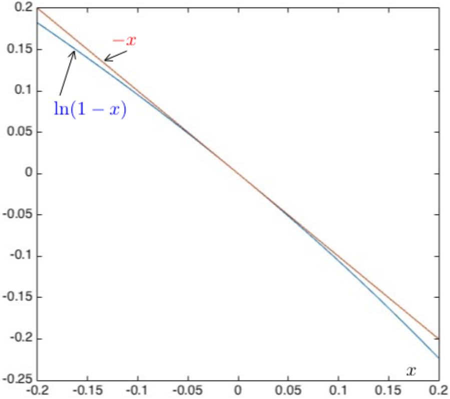

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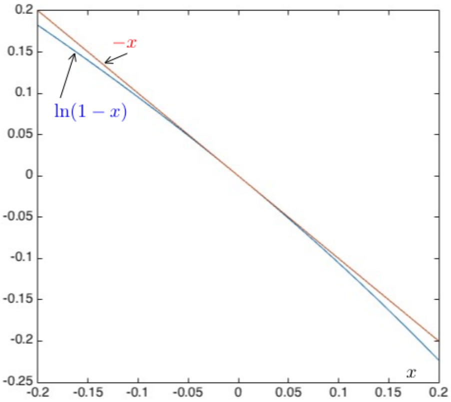

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Hence, $-x \approx \ln(1-x)$ for $|x| \ll 1$.

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Note: $\log_2(x) = \log_2(e) \ln(x) \approx 1.44 \ln(x)$.

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For $p_m = \frac{1}{2}$, we need around $n \ln 2 \approx 0.69n$ boxes.

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Experiment: Choose m cards at random with replacement.

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Plug in and get

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E.g., $n = 10^2 \Rightarrow m = 530; n = 10^3 \Rightarrow m = 7600.$

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