CS70.

1. Random Variables: Brief Review
2. Joint Distributions.
3. Linearity of Expectation

Random Variables: Definitions

Definition

A random variable, X, for a random experiment with sample space Ω is a function $X : \Omega \to \Re$.

Thus, $X(\cdot)$ assigns a real number $X(\omega)$ to each $\omega \in \Omega$.

Definitions

(a) For $a \in \Re$, one defines

$$X^{-1}(a) := \{\omega \in \Omega \mid X(\omega) = a\}.$$

(b) For $A \subset \Re$, one defines

$$X^{-1}(A) := \{\omega \in \Omega \mid X(\omega) \in A\}.$$

(c) The probability that $X = a$ is defined as

$$Pr[X = a] = Pr[X^{-1}(a)].$$

(d) The probability that $X \in A$ is defined as

$$Pr[X \in A] = Pr[X^{-1}(A)].$$

(e) The distribution of a random variable X, is

$$\{(a, Pr[X = a]) : a \in \mathscr{A}\},$$

where $\mathscr{A}$ is the *range* of X. That is, $\mathscr{A} = \{X(\omega), \omega \in \Omega\}$.

Some Distributions.

Binomial Distribution: $B(n,p)$, For $0 \le i \le n$, $Pr[X = i] = \binom{n}{i} p^i (1-p)^{n-i}$. Geometric Distribution: $G(p)$, For $i \ge 1$, $Pr[X = i] = (1-p)^{i-1} p$. Poisson: Next up.

Poisson: Motivation and derivation.

McDonalds: How many arrive at McDonalds in an hour?

Know: average is λ.

What is distribution?

Example: $Pr[2\lambda$ arrivals $]$?

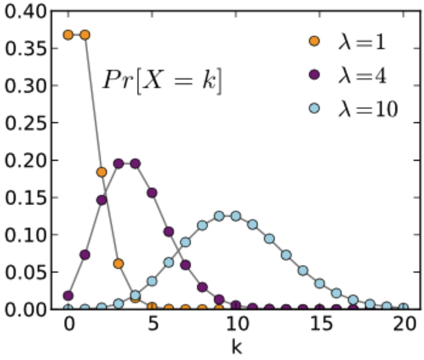
Assumption: "arrivals are independent."

Derivation: cut hour into n intervals of length $1/n$.
$Pr[$ two arrivals $]$ is "$(\lambda/n)^2$" or small if n is large.
Model with binomial.

Poisson

Experiment: flip a coin n times. The coin is such that $Pr[H] = \lambda/n$.
Random Variable: X - number of heads. Thus, $X = B(n, \lambda/n)$.

Poisson Distribution is distribution of X "for large n."



Poisson

Experiment: flip a coin n times. The coin is such that $Pr[H] = \lambda/n$.
Random Variable: X - number of heads. Thus, $X = B(n, \lambda/n)$.
Poisson Distribution is distribution of X "for large n."
We expect $X \ll n$. For $m \ll n$ one has

$$
\begin{aligned}
Pr[X = m] &= \binom{n}{m} p^m (1-p)^{n-m}, \text{ with } p = \lambda/n \\
&= \frac{n(n-1)\cdots(n-m+1)}{m!} \left(\frac{\lambda}{n}\right)^m \left(1 - \frac{\lambda}{n}\right)^{n-m} \\
&= \frac{n(n-1)\cdots(n-m+1)}{n^m} \frac{\lambda^m}{m!} \left(1 - \frac{\lambda}{n}\right)^{n-m} \\
&\approx^{(1)} \frac{\lambda^m}{m!} \left(1 - \frac{\lambda}{n}\right)^{n-m} \approx^{(2)} \frac{\lambda^m}{m!} \left(1 - \frac{\lambda}{n}\right)^{n} \approx^{(3)} \frac{\lambda^m}{m!} e^{-\lambda}.
\end{aligned}
$$

(1) and (2) cuz m is constant and $n \to \infty$; for (3) we used $(1 - a/n)^n \approx e^{-a}$.

Poll:Poisson

A Poisson distribution for parameter λ.

(A) A function on the integers.
(B) $Pr[X = i]$ is the probability of i arrivals.
(C) $Pr[X = i] = \lim_{n\to\infty} Pr[Y = i]$ for $Y = B(n, \lambda/n)$.
(D) Expectation of Poisson is λ.

Expectation - Definition

Definition: The **expected value** (or mean, or expectation) of a random variable X is

$$E[X] = \sum_a a \times Pr[X = a].$$

Theorem:

$$E[X] = \sum_\omega X(\omega) \times Pr[\omega].$$

Proof:

$$\begin{aligned} E[X] &= \sum_a a \times Pr[X = a] \\ &= \sum_a a \times \sum_{\omega : X(\omega) = a} Pr[\omega] \\ &= \sum_a \sum_{\omega : X(\omega) = a} X(\omega) Pr[\omega] \\ &= \sum_\omega X(\omega) Pr[\omega] \end{aligned}$$

□

Poisson Distribution: Definition and Mean

Definition Poisson Distribution with parameter $\lambda > 0$

$$X = P(\lambda) \Leftrightarrow Pr[X = m] = \frac{\lambda^m}{m!} e^{-\lambda}, m \geq 0.$$

Fact: $E[X] = \lambda$.

Proof:

$$\begin{aligned} E[X] &= \sum_{m=1}^{\infty} m \times \frac{\lambda^m}{m!} e^{-\lambda} = e^{-\lambda} \sum_{m=1}^{\infty} \frac{\lambda^m}{(m-1)!} \\ &= e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^{m+1}}{m!} = e^{-\lambda} \lambda \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \\ &= e^{-\lambda} \lambda e^{\lambda} = \lambda. \end{aligned}$$

□

Simeon Poisson

The Poisson distribution is named after:

Siméon Poisson

Siméon Denis Poisson (1781–1840)

Multiple Random Variables.

Experiment: toss two coins. $\Omega = \{HH, TH, HT, TT\}$.

$$X_1(\omega) = \begin{cases} 1, & \text{if coin 1 is heads} \\ 0, & \text{otherwise} \end{cases} \qquad X_2(\omega) = \begin{cases} 1, & \text{if coin 2 is heads} \\ 0, & \text{otherwise} \end{cases}$$

$$X_3(\omega) = \begin{cases} 2, & \text{if both coins are heads} \\ 1, & \text{if exactly one coin is heads} \\ 0, & \text{otherwise} \end{cases}$$

$Pr[X_3] = 2$? 1/4.

$Pr[X_3] = 1$? 1/2.

$Pr[X_3] = 0$? 1/4.

Multiple Random Variables: setup.

Joint Distribution: $\{(a,b,Pr[X=a,Y=b]) : a \in \mathscr{A}, b \in \mathscr{B}\}$, where $\mathscr{A}$ ($\mathscr{B}$) is possible values of X (Y).

$$\sum_{a \in \mathscr{A}, b \in \mathscr{B}} Pr[X=a,Y=b] = 1$$

Marginal for X: $Pr[X=a] = \sum_{b \in \mathscr{B}} Pr[X=a,Y=b]$.
Marginal for Y: $Pr[Y=b] = \sum_{a \in \mathscr{A}} Pr[X=a,Y=b]$.

X/Y	1	2	3	X
1	.2	.1	.1	.4
2	0	0	.3	.3
3	.1	0	.2	.3
Y	.3	.1	.2 .6	

Conditional Probability: $Pr[X=a|Y=b] = \frac{Pr[X=a,Y=b]}{Pr[Y=b]}$.

Joint Distribution: two coins.

Experiment: toss two coins. $\Omega = \{HH, TH, HT, TT\}$.

$$X(\omega) = \begin{cases} 1, & \text{if coin 1 is heads} \\ 0, & \text{otherwise} \end{cases} \quad Y(\omega) = \begin{cases} 2, & \text{if both coins are heads} \\ 1, & \text{if exactly one coin is heads} \\ 0, & \text{otherwise} \end{cases}$$

X/Y	0	1	2	X
0	.25	.25	0	.5
1	0	.25	.25	.5
Y	.25	.5	.25	

Review: Independence of Events

- Events A, B are independent if $Pr[A \cap B] = Pr[A]Pr[B]$.
- Events A, B, C are mutually independent if

  A, B are independent, A, C are independent, B, C are independent

  and $Pr[A \cap B \cap C] = Pr[A]Pr[B]Pr[C]$.
- Events $\{A_n, n \geq 0\}$ are mutually independent if ....
- Example: $X, Y \in \{0, 1\}$ two fair coin flips $\Rightarrow X, Y, X \oplus Y$ are pairwise independent but not mutually independent.
- Example: $X, Y, Z \in \{0, 1\}$ three fair coin flips are mutually independent.

Independent Random Variables.

Definition: Independence

The random variables X and Y are **independent** if and only if

$$Pr[Y = b | X = a] = Pr[Y = b], \text{ for all } a \text{ and } b.$$

Fact:

X, Y are independent if and only if

$$Pr[X = a, Y = b] = Pr[X = a]Pr[Y = b], \text{ for all } a \text{ and } b.$$

Follows from $Pr[A \cap B] = Pr[A|B]Pr[B]$ (Product rule.)

Independence: Examples

Example 1
Roll two die. X, Y = number of pips on the two dice. X, Y are independent.

Indeed: $Pr[X = a, Y = b] = \frac{1}{36}, Pr[X = a] = Pr[Y = b] = \frac{1}{6}$.

Example 2
Roll two die. X = total number of pips, Y = number of pips on die 1 minus number on die 2. X and Y are not independent.

Indeed: $Pr[X = 12, Y = 1] = 0 \neq Pr[X = 12]Pr[Y = 1] > 0$.

Example 3
Flip a fair coin five times, X = number of Hs in first three flips, Y = number of Hs in last two flips. X and Y are independent.

Indeed:

$$Pr[X = a, Y = b] = \binom{3}{a}\binom{2}{b}2^{-5} = \binom{3}{a}2^{-3} \times \binom{2}{b}2^{-2} = Pr[X = a]Pr[Y = b].$$

Joint Distribution: non-indepence.

Experiment: toss two coins. $\Omega = \{HH, TH, HT, TT\}$.

$$X(\omega) = \begin{cases} 1, & \text{if coin 1 is heads} \\ 0, & \text{otherwise} \end{cases} \qquad Y(\omega) = \begin{cases} 2, & \text{if both coins are heads} \\ 1, & \text{if exactly one coin is he} \\ 0, & \text{otherwise} \end{cases}$$

X/Y	0	1	2	X
0	.25	.25	0	.5
1	0	.25	.25	.5
Y	.25	.5	.25	

$Pr[Y = 1] =?$ .5

$Pr[Y = 1|X = 1] =?$ .25/.5 = .5

$Pr[Y = 2] =?$ .25

$Pr[Y = 2|X = 1] =?$ $.25/.5$ = $.5 \neq .25 = Pr[Y = 2]$

Not independent. All events should be independent.

Linearity of Expectation

Theorem:

$$E[X+Y] = E[X] + E[Y]$$

$$E[cX] = cE[X]$$

Proof: $E[X] = \sum_{\omega \in \Omega} X(\omega) \times P[\omega]$.

$$\begin{aligned} E[X+Y] &= \sum_{\omega \in \Omega} (X(\omega) + Y(\omega)) Pr[\omega] \\ &= \sum_{\omega \in \Omega} X(\omega) Pr[\omega] + Y(\omega) Pr[\omega] \\ &= \sum_{\omega \in \Omega} X(\omega) Pr[\omega] + \sum_{\omega \in \Omega} Y(\omega) Pr[\omega] \\ &= E[X] + E[Y] \end{aligned}$$

Linearity of Expectation

Theorem: Expectation is linear

$$E[a_1 X_1 + \cdots + a_n X_n] = a_1 E[X_1] + \cdots + a_n E[X_n].$$

Proof:

$$\begin{aligned}
&E[a_1 X_1 + \cdots + a_n X_n] \\
&\quad = \sum_\omega (a_1 X_1 + \cdots + a_n X_n)(\omega) Pr[\omega] \\
&\quad = \sum_\omega (a_1 X_1(\omega) + \cdots + a_n X_n(\omega)) Pr[\omega] \\
&\quad = a_1 \sum_\omega X_1(\omega) Pr[\omega] + \cdots + a_n \sum_\omega X_n(\omega) Pr[\omega] \\
&\quad = a_1 E[X_1] + \cdots + a_n E[X_n].
\end{aligned}$$

□

Note: If we set $Y = a_1 X_1 + \cdots + a_n X_n$ and use the distibution, $E[Y] = \sum_y y Pr[Y = y]$, we have some trouble! Summing over sample space was easier.

Using Linearity - 1: Pips (dots) on dice

Roll a die n times.

X_m = number of pips on roll m.

$X = X_1 + \cdots + X_n$ = total number of pips in n rolls.

$$
\begin{aligned}
E[X] &= E[X_1 + \cdots + X_n] \\
&= E[X_1] + \cdots + E[X_n], \text{ by linearity} \\
&= nE[X_1], \text{ because the } X_m \text{ have the same distribution}
\end{aligned}
$$

Now,

$$E[X_1] = 1 \times \frac{1}{6} + \cdots + 6 \times \frac{1}{6} = \frac{6 \times 7}{2} \times \frac{1}{6} = \frac{7}{2}.$$

Hence,

$$E[X] = \frac{7n}{2}.$$

Note: Computing $\sum_x xPr[X = x]$ directly is not easy!

Indicators

Definition
Let A be an event. The random variable X defined by

$$X(\omega) = \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{if } \omega \notin A \end{cases}$$

is called the indicator of the event A.

Note that $Pr[X = 1] = Pr[A]$ and $Pr[X = 0] = 1 - Pr[A]$.

Hence,

$$E[X] = 1 \times Pr[X = 1] + 0 \times Pr[X = 0] = Pr[A].$$

This random variable $X(\omega)$ is sometimes written as

$$1\{\omega \in A\} \text{ or } 1_A(\omega).$$

Thus, we will write $X = 1_A$.

Using Linearity - 2: Fixed point.

Hand out assignments at random to n students.

$X =$ number of students that get their own assignment back.

What is $Pr[X = m]$? Tricky ....

$X = X_1 + \cdots + X_n$ where

$X_m = 1\{\text{student } m \text{ gets their own assignment back}\}$.

$Pr[X_i = 1]$? $\frac{1}{n}$. Student is equally likely to get any assignment.

$$\begin{aligned} E[X] &= E[X_1 + \cdots + X_n] \\ &= E[X_1] + \cdots + E[X_n], \text{ by linearity} \\ &= nE[X_1], \text{ because all the } X_m \text{ have the same distribution} \\ &= nPr[X_1 = 1], \text{ because } X_1 \text{ is an indicator} \\ &= n(1/n), \text{ because student 1 is equally likely} \\ &\qquad\qquad \text{to get any one of the } n \text{ assignments} \\ &= 1. \end{aligned}$$

Note:linearity holds even though the X_m are not independent.

Using Linearity - 3: Binomial Distribution.

Flip n coins with heads probability p. X - number of heads

Binomial Distibution: $Pr[X = i]$, for each i.

$$Pr[X = i] = \binom{n}{i} p^i (1-p)^{n-i}.$$

$$E[X] = \sum_i i \times Pr[X = i] = \sum_i i \times \binom{n}{i} p^i (1-p)^{n-i}.$$

Uh oh. ... Or... a better approach: Let

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$E[X_i] = 1 \times Pr[\text{“}heads''] + 0 \times Pr[\text{“}tails''] = p.$

Moreover $X = X_1 + \cdots X_n$ and

$E[X] = E[X_1] + E[X_2] + \cdots E[X_n] = n \times E[X_i] = np.$

Using Linearity - 4

Assume A and B are disjoint events. Then $1_{A\cup B}(\omega) = 1_A(\omega) + 1_B(\omega)$.

Taking expectation, we get

$$Pr[A\cup B] = E[1_{A\cup B}] = E[1_A + 1_B] = E[1_A] + E[1_B] = Pr[A] + Pr[B].$$

In general, $1_{A\cup B}(\omega) = 1_A(\omega) + 1_B(\omega) - 1_{A\cap B}(\omega)$.

Taking expectation, we get $Pr[A\cup B] = Pr[A] + Pr[B] - Pr[A\cap B]$.

Observe that if $Y(\omega) = b$ for all ω, then $E[Y] = b$.

Thus, $E[X+b] = E[X] + b$.

Empty Bins

Experiment: Throw m balls into n bins.

Y - number of empty bins.

Distribution is horrible.

Expectation? X_i - indicator for bin i being empty.

$$Y = X_1 + \cdots X_n.$$

$Pr[X_1 = 1] = (1 - \frac{1}{n})^m$. $\rightarrow$ $E[Y] = n(1 - \frac{1}{n})^m$.

For $n = m$ and large n, $(1 - 1/n)^n \approx \frac{1}{e}$.

$\frac{n}{e} \approx 0.368n$ empty bins on average.

Coupon Collectors Problem.

Experiment: Get random coupon from n until get all n coupons.
Outcomes: $\{123145..., 56765...\}$
Random Variable: X - length of outcome.

Today: $E[X]$?

Geometric Distribution: Expectation

$$X =_D G(p), \text{ i.e., } Pr[X = n] = (1-p)^{n-1}p, n \geq 1.$$

One has

$$E[X] = \sum_{n=1}^{\infty} nPr[X = n] = \sum_{n=1}^{\infty} n(1-p)^{n-1}p.$$

Thus,

$$\begin{aligned}
E[X] &= p + 2(1-p)p + 3(1-p)^2p + 4(1-p)^3p + \cdots \\
(1-p)E[X] &= \quad (1-p)p + 2(1-p)^2p + 3(1-p)^3p + \cdots \\
pE[X] &= p + \ (1-p)p + \ (1-p)^2p + \ (1-p)^3p + \cdots \\
&\quad \text{by subtracting the previous two identities} \\
&= \sum_{n=1}^{\infty} Pr[X = n] = 1.
\end{aligned}$$

Hence,

$$E[X] = \frac{1}{p}.$$

Time to collect coupons

X-time to get n coupons.

X_1 - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.

X_2 - time to get second coupon after getting first.

Pr[“get second coupon”|“got ~~milk~~ first coupon”] $= \frac{n-1}{n}$

$E[X_2]$? Geometric ! ! ! $\implies E[X_2] = \frac{1}{p} = \frac{1}{\frac{n-1}{n}} = \frac{n}{n-1}$.

Pr[“getting ith coupon|“got $i-1$rst coupons”] $= \frac{n-(i-1)}{n} = \frac{n-i+1}{n}$

$E[X_i] = \frac{1}{p} = \frac{n}{n-i+1}, i = 1, 2, \ldots, n.$

$$\begin{aligned} E[X] &= E[X_1] + \cdots + E[X_n] = \frac{n}{n} + \frac{n}{n-1} + \frac{n}{n-2} + \cdots + \frac{n}{1} \\ &= n(1 + \frac{1}{2} + \cdots + \frac{1}{n}) =: nH(n) \approx n(\ln n + \gamma) \end{aligned}$$

Coupons: Poll

Collect n coupons!

What's True?

(A) $X_1 = \frac{n}{n} = 1$. No. Its an integer.
(B) $X_2 = \frac{n}{n-1}$. No. Random variable.
(C) $Pr[\text{getting second}|\text{got first}] = \frac{n-1}{n}$ Yes.
(D) $E[X_2] = \frac{n}{n-1}$. $E[G(p)]$ for $p = n - 1/n$
(E) $E[X_n] = n$. $E[G(p)]$ for $p = 1/n$.
(F) $\sum_i E[X_i] = \sum_{i=0}^{n-1} \frac{n}{n-i}$ Yes.
(G) $\sum_i E[X_i] = n \sum_{i=1}^{n} \frac{1}{n}$ Factor out n and change index.

Review: Harmonic sum

$$H(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \int_1^n \frac{1}{x} dx = \ln(n).$$

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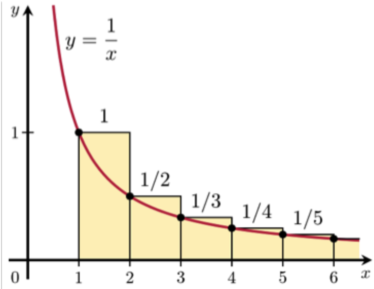
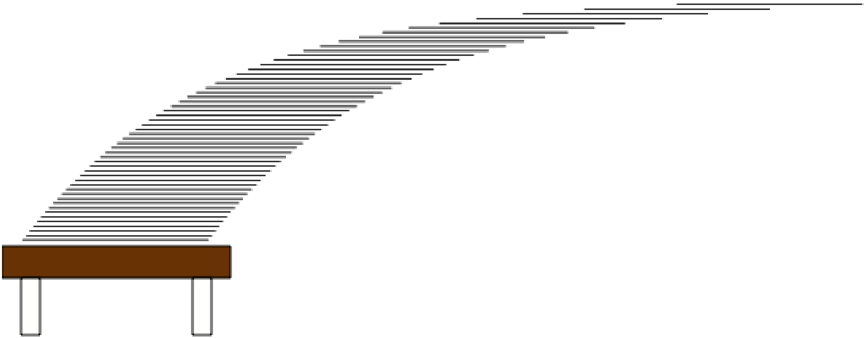


A good approximation is

$H(n) \approx \ln(n) + \gamma$ where $\gamma \approx 0.58$ (Euler-Mascheroni constant).

Harmonic sum: Paradox

Consider this stack of cards (no glue!):



If each card has length 2, the stack can extend $H(n)$ to the right of the table. As n increases, you can go as far as you want!

Paradox

par·a·dox

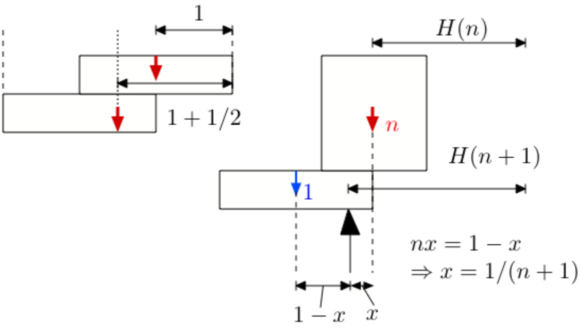
/ˈperəˌdäks/

noun

a statement or proposition that, despite sound (or apparently sound) reasoning from acceptable premises, leads to a conclusion that seems senseless, logically unacceptable, or self-contradictory.
"a potentially serious conflict between quantum mechanics and the general theory of relativity known as the information paradox"

- a seemingly absurd or self-contradictory statement or proposition that when investigated or explained may prove to be well founded or true.
  "in a paradox, he has discovered that stepping back from his job has increased the rewards he gleans from it"
  synonyms: contradiction, contradiction in terms, self-contradiction, inconsistency, incongruity; More
- a situation, person, or thing that combines contradictory features or qualities.
  "the mingling of deciduous trees with elements of desert flora forms a fascinating ecological paradox"

Stacking



The cards have width 2. Induction shows that the center of gravity after n cards is $H(n)$ away from the right-most edge.
Video.

Calculating $E[g(X)]$

Let $Y = g(X)$. Assume that we know the distribution of X.

We want to calculate $E[Y]$.

Method 1: We calculate the distribution of Y:

$$Pr[Y = y] = Pr[X \in g^{-1}(y)] \text{ where } g^{-1}(x) = \{x \in \Re : g(x) = y\}.$$

This is typically rather tedious!

Method 2: We use the following result.

Theorem:

$$E[g(X)] = \sum_x g(x) Pr[X = x].$$

Proof:

$$\begin{aligned}
E[g(X)] &= \sum_\omega g(X(\omega)) Pr[\omega] = \sum_x \sum_{\omega \in X^{-1}(x)} g(X(\omega)) Pr[\omega] \\
&= \sum_x \sum_{\omega \in X^{-1}(x)} g(x) Pr[\omega] = \sum_x g(x) \sum_{\omega \in X^{-1}(x)} Pr[\omega] \\
&= \sum_x g(x) Pr[X = x].
\end{aligned}$$

$\square$

An Example

Let X be uniform in $\{-2,-1,0,1,2,3\}$.

Let also $g(X) = X^2$. Then (method 2)

$$\begin{aligned} E[g(X)] &= \sum_{x=-2}^{3} x^2 \times \frac{1}{6} \\ &= \{4+1+0+1+4+9\} \times \frac{1}{6} = \frac{19}{6}. \end{aligned}$$

Method 1 - We find the distribution of $Y = X^2$:

$$Y = \begin{cases} 4, & \text{w.p. } \frac{2}{6} \\ 1, & \text{w.p. } \frac{2}{6} \\ 0, & \text{w.p. } \frac{1}{6} \\ 9, & \text{w.p. } \frac{1}{6}. \end{cases}$$

Thus,

$$E[Y] = 4 \times \frac{2}{6} + 1 \times \frac{2}{6} + 0 \times \frac{1}{6} + 9 \times \frac{1}{6} = \frac{19}{6}.$$

Summary

Probability Space: Ω, $Pr[\omega] \geq 0$, $\sum_{\omega} Pr[\omega] = 1$.

Random Variable: Function on Sample Space.

Distribution: Function $Pr[X = a] \geq 0$. $\sum_{a} Pr[X = a] = 1$.

Expectation: $E[X] = \sum_{\omega} Pr[\omega] = \sum_{a} Pr[X = a]$.

Many Random Variables: each one function on a sample space.

Joint Distributions: Function $Pr[X = a, Y = b] \geq 0$.
$\sum_{a,b} Pr[X = a, Y = b] = 1$.

Linearity of Expectation: $E[X + Y] = E[X] + E[Y]$.

Applications: compute expectations by decomposing.

Indicators: Empty bins, Fixed points.
Time to Coupon: Sum times to "next" coupon.

$Y = f(X)$ is Random Variable.
Distribution of Y from distribution of X.