CS70.

1. Random Variables: Brief Review
2. Joint Distributions.
3. Linearity of Expectation

Random Variables: Definitions

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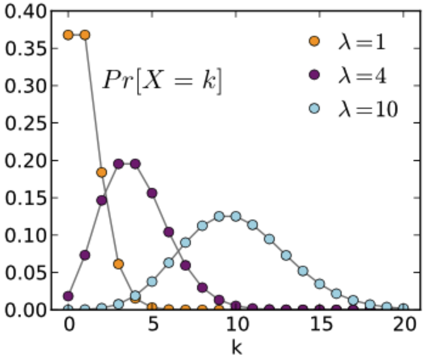

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$\square$

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□

Simeon Poisson

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$$X_1(\omega) = \begin{cases} 1, & \text{if coin 1 is heads} \\ 0, & \text{otherwise} \end{cases} \qquad X_2(\omega) = \begin{cases} 1, & \text{if coin 2 is heads} \\ 0, & \text{otherwise} \end{cases}$$

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X/Y	1	2	3	X
1	.2	.1	.1	.4
2	0	0	.3	.3
3	.1	0	.2	.3
Y	.3	.1	.2	

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Follows from $Pr[A \cap B] = Pr[A|B]Pr[B]$ (Product rule.)

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Joint Distribution: non-indepence.

Experiment: toss two coins. $\Omega = \{HH, TH, HT, TT\}$.

$$X(\omega) = \begin{cases} 1, & \text{if coin 1 is heads} \\ 0, & \text{otherwise} \end{cases} \qquad Y(\omega) = \begin{cases} 2, & \text{if both coins are heads} \\ 1, & \text{if exactly one coin is he} \\ 0, & \text{otherwise} \end{cases}$$

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Not independent. All events should be independent.

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Theorem:

$$E[X+Y] = E[X] + E[Y]$$

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$$\begin{aligned}
E[X+Y] &= \sum_{\omega\in\Omega}(X(\omega)+Y(\omega))Pr[\omega] \\
&= \sum_{\omega\in\Omega} X(\omega)Pr[\omega] + Y(\omega)Pr[\omega] \\
&= \sum_{\omega\in\Omega} X(\omega)Pr[\omega] + \sum_{\omega\in\Omega} Y(\omega)Pr[\omega] \\
&= E[X] + E[Y]
\end{aligned}$$

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Note: If we set $Y = a_1 X_1 + \cdots + a_n X_n$ and use the distibution, $E[Y] = \sum_y y Pr[Y = y]$, we have some trouble! Summing over sample space was easier.

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Roll a die n times.

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Note: Computing $\sum_x xPr[X = x]$ directly is not easy!

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Thus, we will write $X = 1_A$.

Using Linearity - 2: Fixed point.

Hand out assignments at random to n students.

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Note:linearity holds even though the X_m are not independent.

Using Linearity - 2: Fixed point.

Hand out assignments at random to n students.

$X =$ number of students that get their own assignment back.

What is $Pr[X = m]$? Tricky ....

$X = X_1 + \cdots + X_n$ where
$X_m = 1\{$student m gets their own assignment back$\}$.

$Pr[X_i = 1]$? $\frac{1}{n}$. Student is equally likely to get any assignment.

$$\begin{aligned}
E[X] &= E[X_1 + \cdots + X_n] \\
&= E[X_1] + \cdots + E[X_n], \text{ by linearity} \\
&= nE[X_1], \text{ because all the } X_m \text{ have the same distribution} \\
&= nPr[X_1 = 1], \text{ because } X_1 \text{ is an indicator} \\
&= n(1/n), \text{ because student 1 is equally likely} \\
&\qquad\qquad \text{to get any one of the } n \text{ assignments} \\
&= 1.
\end{aligned}$$

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Using Linearity - 3: Binomial Distribution.

Flip n coins with heads probability p.

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Thus, $E[X+b] = E[X] + b$.

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Experiment: Throw m balls into n bins.

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$\frac{n}{e} \approx 0.368n$ empty bins on average.

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$E[X_2]$? Geometric ! ! ! $\implies E[X_2] = \frac{1}{p} = \frac{1}{\frac{n-1}{n}} = \frac{n}{n-1}$.

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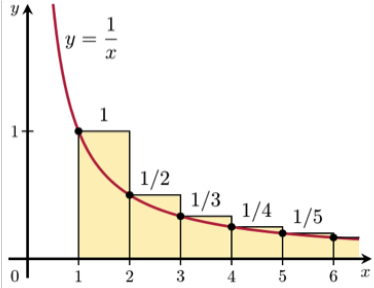

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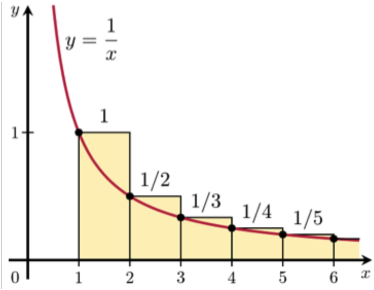

A good approximation is

$H(n) \approx \ln(n) + \gamma$ where $\gamma \approx 0.58$ (Euler-Mascheroni constant).

Harmonic sum: Paradox

Consider this stack of cards (no glue!):

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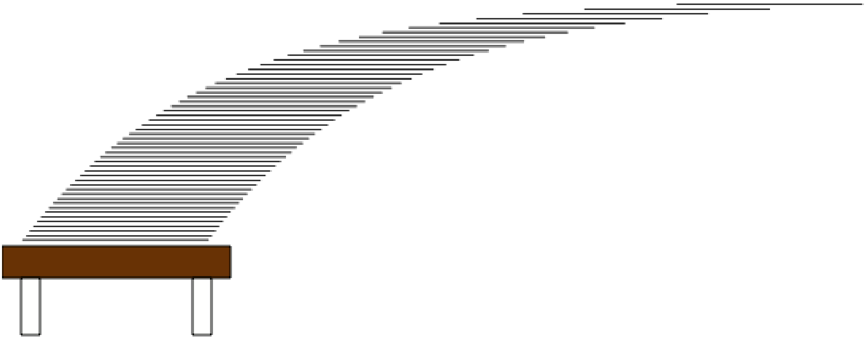

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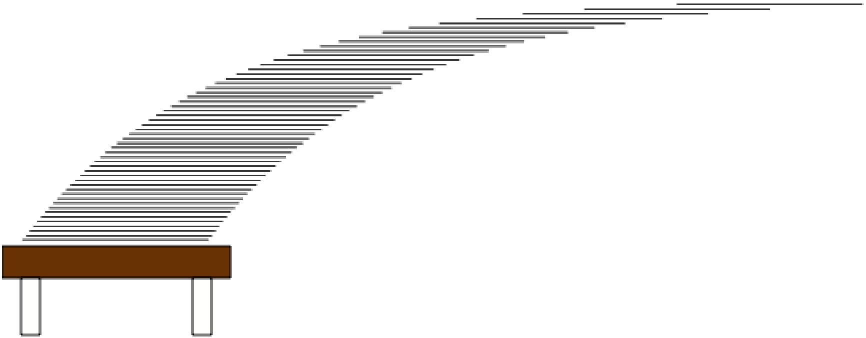

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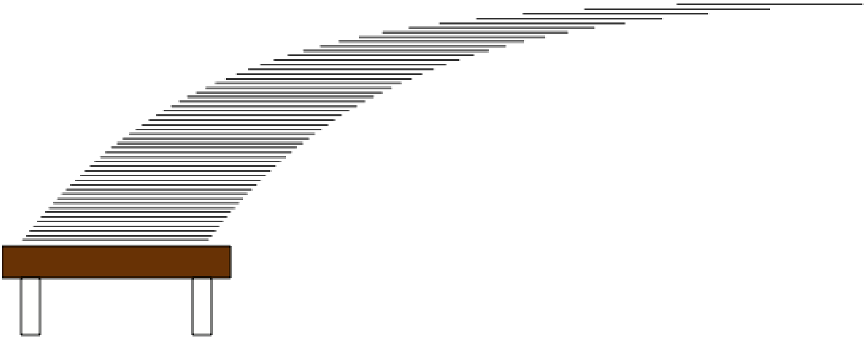


If each card has length 2, the stack can extend $H(n)$ to the right of the table. As n increases, you can go as far as you want!

Paradox

par·a·dox

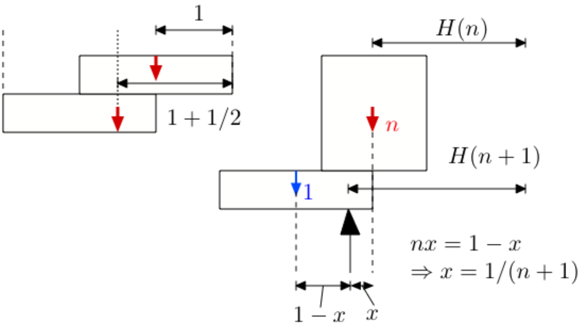
/ˈperəˌdäks/

noun

a statement or proposition that, despite sound (or apparently sound) reasoning from acceptable premises, leads to a conclusion that seems senseless, logically unacceptable, or self-contradictory.
"a potentially serious conflict between quantum mechanics and the general theory of relativity known as the information paradox"

- a seemingly absurd or self-contradictory statement or proposition that when investigated or explained may prove to be well founded or true.
  "in a paradox, he has discovered that stepping back from his job has increased the rewards he gleans from it"
  synonyms: contradiction, contradiction in terms, self-contradiction, inconsistency, incongruity; More
- a situation, person, or thing that combines contradictory features or qualities.
  "the mingling of deciduous trees with elements of desert flora forms a fascinating ecological paradox"

Stacking



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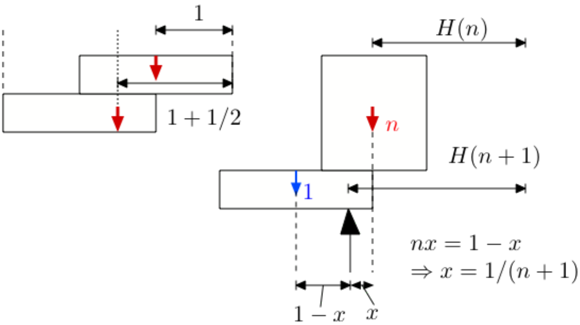

The cards have width 2.

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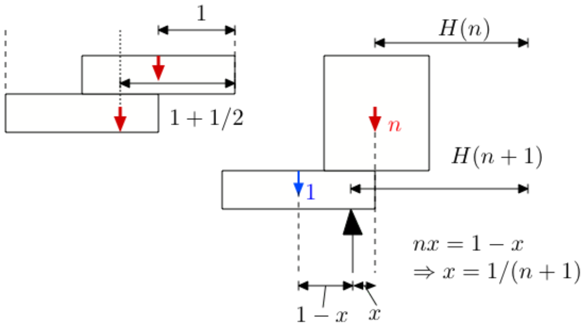

The cards have width 2. Induction shows that the center of gravity after n cards is $H(n)$ away from the right-most edge.
Video.

Calculating $E[g(X)]$

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$\square$

An Example

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Let X be uniform in $\{-2, -1, 0, 1, 2, 3\}$.

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Let also $g(X) = X^2$.

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