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Also, prediction from evidence.

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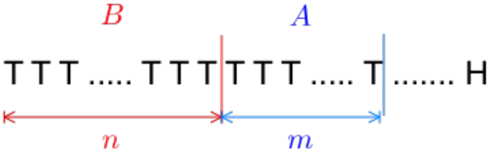

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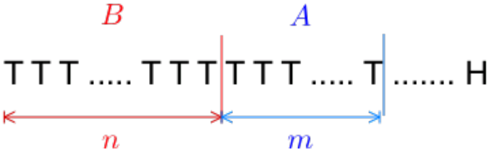

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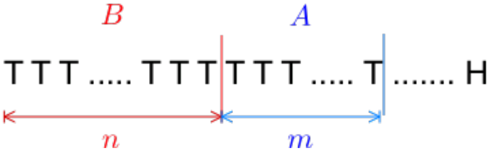

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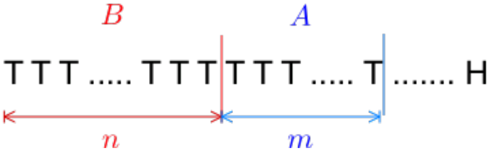

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The coin is memoryless , therefore, so is X.
Independent coin: $Pr[H|\text{'any previous set of coin tosses'}] = p$

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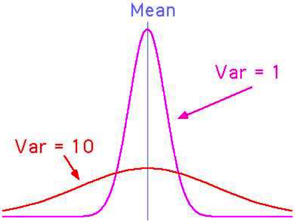

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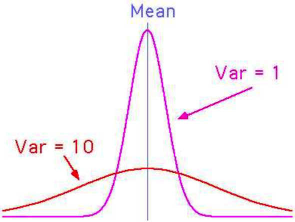

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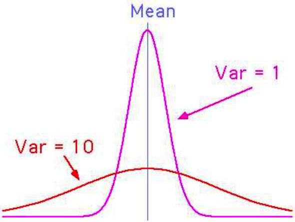

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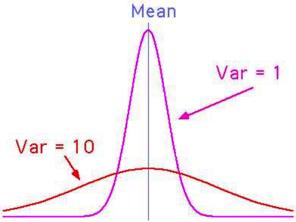

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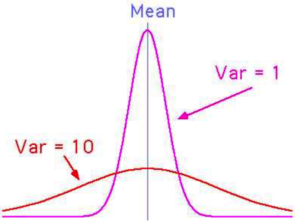

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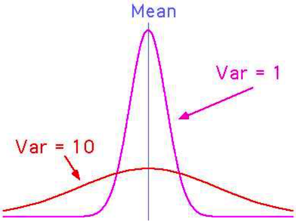

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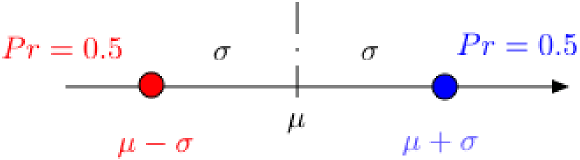

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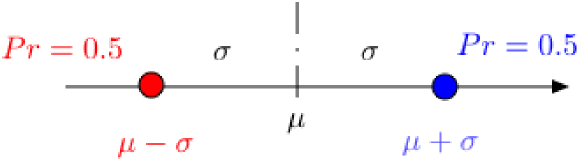

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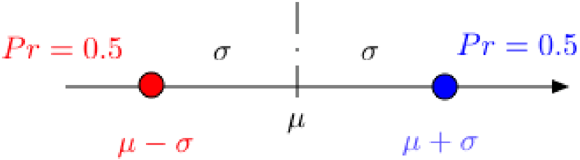

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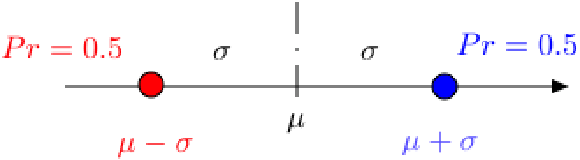

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$$var(X) = \sigma^2 \text{ and } \sigma(X) = \sigma.$$

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Keep expectation small using $1/M$.
Yields $2E[X]$ for $E[|X - E[X]|]$, and $\approx \sqrt{M}$ for $\approx E[\sqrt{X - E(X)}]$.

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This gives

$$var(X) = \frac{1+3n+2n^2}{6} - \frac{(n+1)^2}{4} = \frac{n^2-1}{12}.$$

(Sort of $\int_0^{1/2} x^2 dx = \frac{x^3}{3}$.)

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$$= \frac{1}{n}$$

Fixed points.

Number of fixed points in a random permutation of n items.
"Number of student that get homework back."

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$Var(X) = E(X^2) - (E(X))^2 = 2 - 1 = 1.$

Poll: fixed points.

What's true?

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Yes. $1^2 = 1$ and $0^2 = 1$, $X_i \in \{0, 1\}$.

Variance: binomial.

$$E[X^2] = \sum_{i=0}^{n} i^2 \times \binom{n}{i} p^i (1-p)^{n-i}.$$

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Too hard!

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Maybe not much easier...

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Maybe not much easier...but there is a payoff.

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□

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\begin{aligned}
E[XY] &= \sum_a \sum_b a \times b \times Pr[X = a, Y = b] \\
&= \sum_a \sum_b a \times b \times Pr[X = a]Pr[Y = b] \\
&= (\sum_a aPr[X = a])(\sum_b bPr[Y = b]) \\
&= E[X]E[Y]
\end{aligned}
$$

Variance of sum of two independent random variables

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If $X, Y, Z, \ldots$ are pairwise independent, then

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$$E[XY] = E[X]E[Y] = 0. \text{ Also, } E[XZ] = E[YZ] = \cdots = 0.$$

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If $X, Y, Z, \ldots$ are pairwise independent, then

$$var(X + Y + Z + \cdots) = var(X) + var(Y) + var(Z) + \cdots .$$

Proof:
Since shifting the random variables does not change their variance, let us subtract their means.

That is, we assume that $E[X] = E[Y] = \cdots = 0$.

Then, by independence,

$$E[XY] = E[X]E[Y] = 0. \text{ Also, } E[XZ] = E[YZ] = \cdots = 0.$$

Hence,

$$\begin{aligned} var(X + Y + Z + \cdots) &= E((X + Y + Z + \cdots)^2) \\ &= E(X^2 + Y^2 + Z^2 + \cdots + 2XY + 2XZ + 2YZ + \cdots) \end{aligned}$$

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&= var(X) + var(Y) + var(Z) + \cdots .
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$\square$

Variance of Binomial Distribution.

Flip coin with heads probability p.

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Poisson Distribution: Variance.

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$$\begin{aligned} E[(X-E[X])(Y-E[Y])] &= E[XY - E[X]Y - XE[Y] + E[X]E[Y]] \\ &= E[XY] - E[X]E[Y] - E[X]E[Y] + E[X]E[Y] \\ &= E[XY] - E[X]E[Y]. \end{aligned}$$

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Definition The correlation of X, Y, $Cor(X, Y)$ is

$$corr(X, Y) : \frac{cov(X, Y)}{\sigma(X)\sigma(Y)}.$$

Theorem: $-1 \leq corr(X, Y) \leq 1$.
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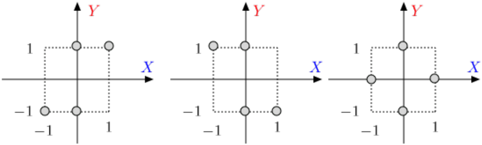
□

Shifting and scaling doesn't change correlation.

Examples of Covariance

Four equally likely pairs of values



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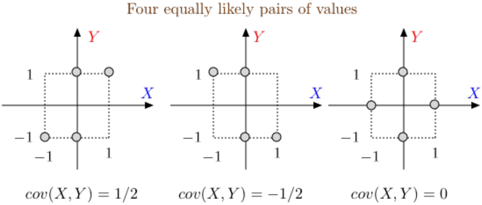

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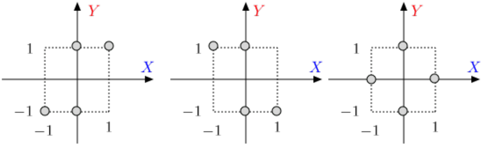

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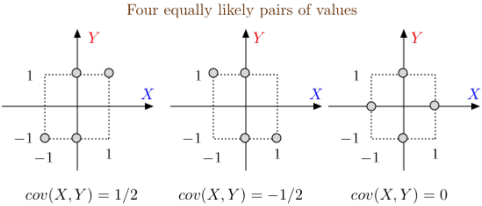

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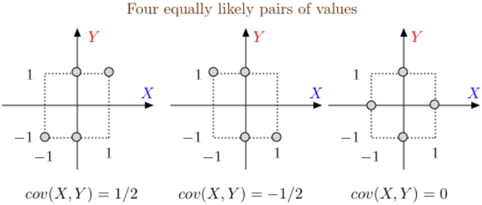

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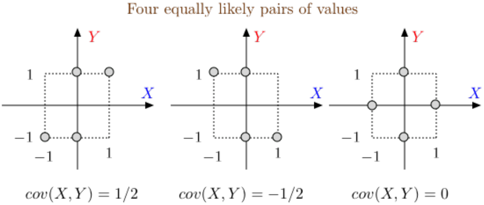

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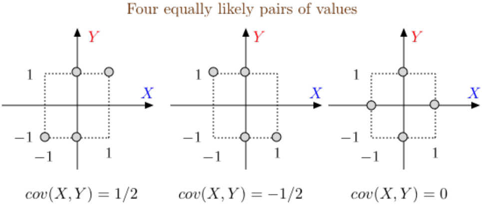

Note that $E[X] = 0$ and $E[Y] = 0$ in these examples. Then $cov(X, Y) = E[XY]$.

When $cov(X, Y) > 0$, the RVs X and Y tend to be large or small together. X and Y are said to be positively correlated.

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When $cov(X, Y) = 0$, we say that X and Y are uncorrelated.

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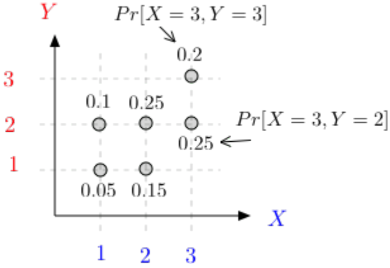

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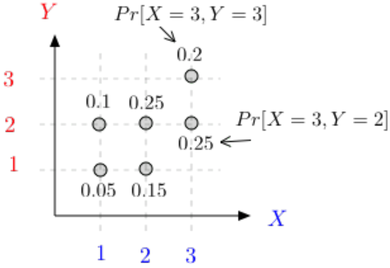

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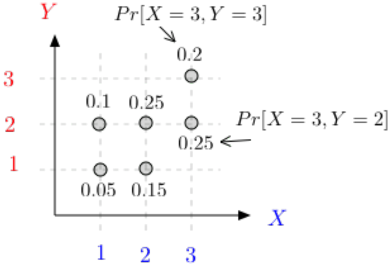

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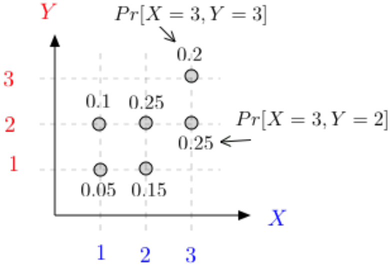

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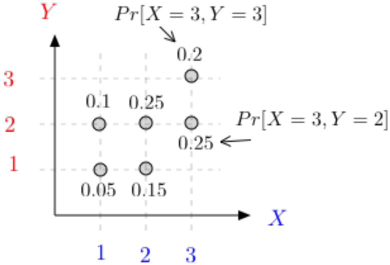

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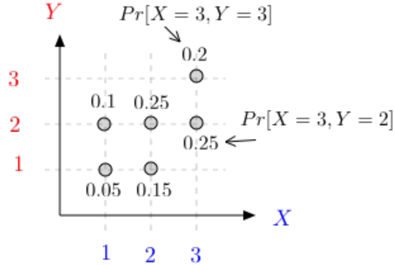

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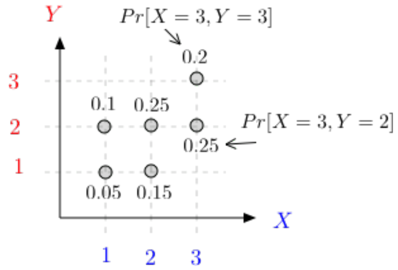

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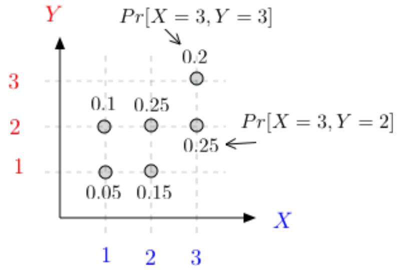

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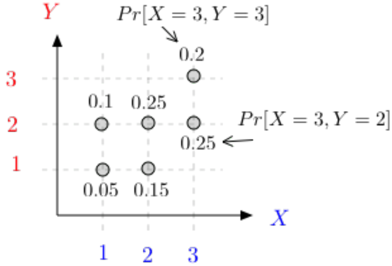

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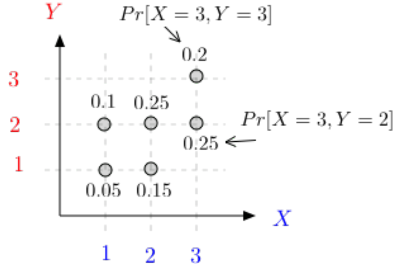

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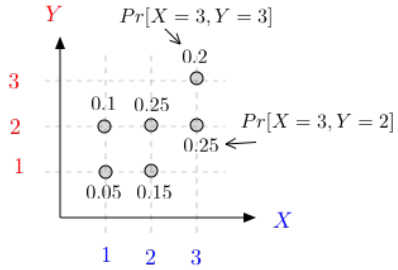

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$\square$

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True. Otherwise average would be higher.

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Assume $f : \Re \to [0, \infty)$ is nondecreasing. Then,

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If $X \geq a$, it reads $1 \leq f(x)/f(a)$, since $f(\cdot)$ is nondecreasing.

Taking the expectation yields the inequality,
expectation of an indicator is the probability.
and expectation is monotone, e.g., weighted sum of points.

Markov's inequality

The inequality is named for Andrey Markov, though in work by Pafnuty Chebyshev. (Sometimes) called Chebyshev's first inequality.

Theorem Markov's Inequality
Assume $f : \Re \to [0, \infty)$ is nondecreasing. Then,

$$Pr[X \geq a] \leq \frac{E[f(X)]}{f(a)}, \text{ for all } a \text{ such that } f(a) > 0.$$

Proof:

Claim:

$$1\{X \geq a\} \leq \frac{f(X)}{f(a)}.$$

If $X < a$, the inequality reads $0 \leq f(x)/f(a)$, since $f(\cdot) \geq 0$.
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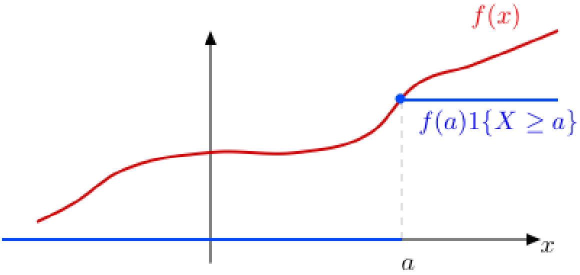
That is, $\sum_v Pr[X = v] 1\{v \geq a\} \leq \sum_v Pr[X = v] \frac{f(v)}{f(a)}$.

□

Intuition: $E[f(X)] \geq f(a) Pr[X > a] = f(a) Pr[X > f(a)]$.

A picture



$$f(a)1\{X \geq a\} \leq f(x) \Rightarrow 1\{X \geq a\} \leq \frac{f(X)}{f(a)}$$

$$\Rightarrow Pr[X \geq a] \leq \frac{E[f(X)]}{f(a)}$$

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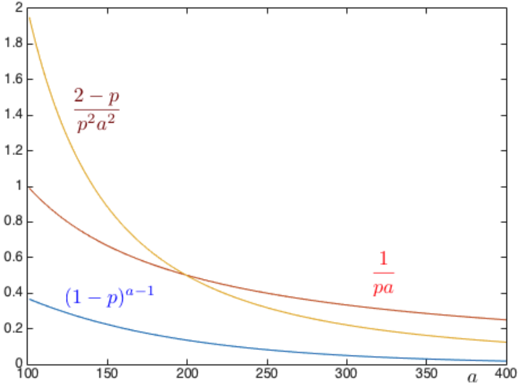

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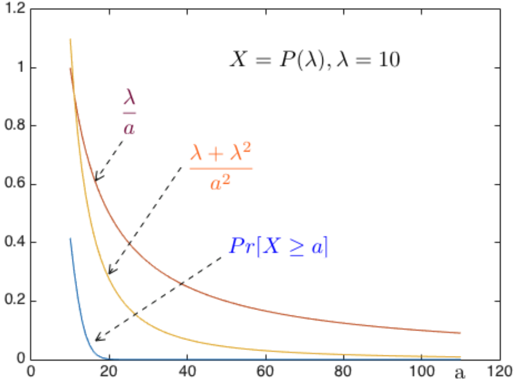

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This result confirms that the variance measures the "deviations from the mean."

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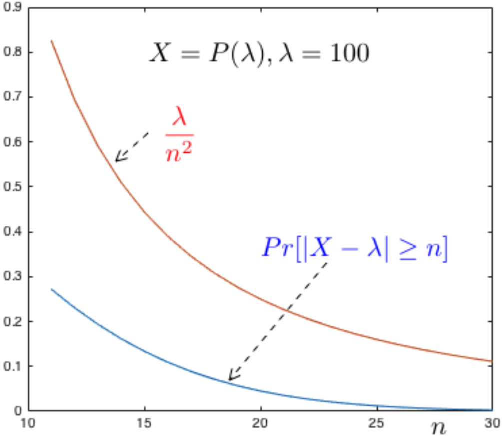

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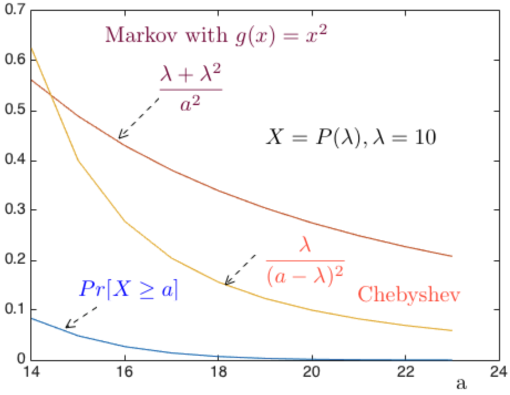

Estimation: Expectation and Mean Squared Error.

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More on Tuesday.

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