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$E[Y|X]$ denotes a function $f(x) = E[Y|X = x]$

Estimation: linear regression.

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Thus: $LLSE[Y|X] = E[Y] + \frac{Cov(X,Y)}{Var(X)}(X - E(X))$.

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Observing X reduces variance by fraction of $corr(X,Y) = \frac{cov(X,Y)^2}{var(X)var(Y)}$.

By a factor $1 - corr(X,Y)$. Correlation Coefficient is fraction of explained variance.

Poll

Correlation coefficient of 1, what is $E[(Y - L[Y|X])^2]$?

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If $Y = X + B(n,p)$, and $X = B(n,p)$, what is Correlation Coefficient?

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$1/2$. Half the variance of Y is explained by X.

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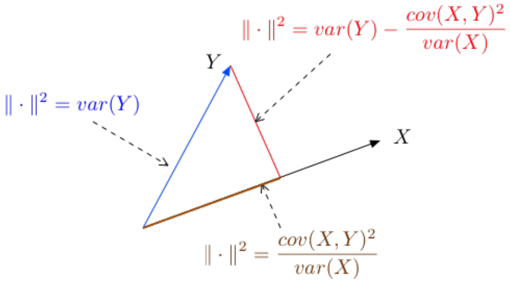

Vector Y at dimension ω is $\frac{1}{\sqrt{\Omega}} Y(\omega)$

Linear Regression Examples

Example:

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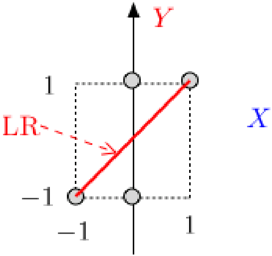

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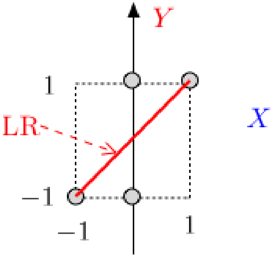

We find:

$$E[X] =$$

Linear Regression Examples

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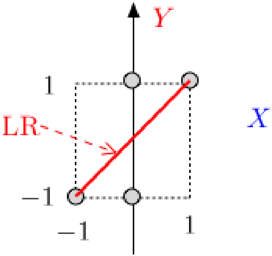

We find:

$E[X] = 0;$

Linear Regression Examples

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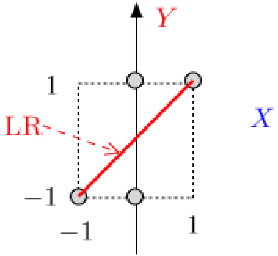

We find:

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Linear Regression Examples

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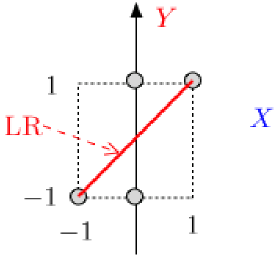

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Linear Regression Examples

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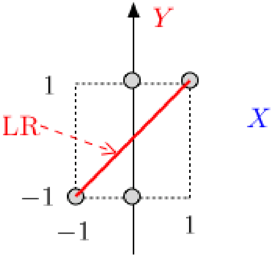

We find:

$$E[X] = 0; E[Y] = 0; E[X^2] =$$

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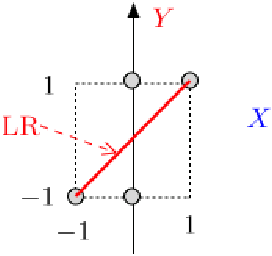

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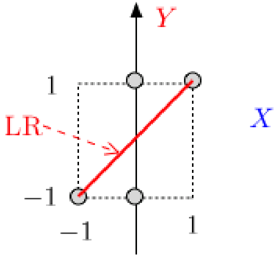

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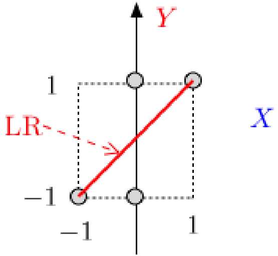

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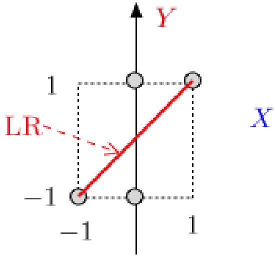

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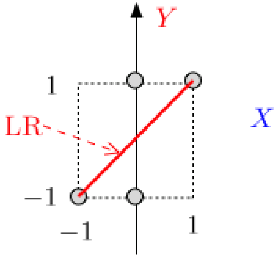

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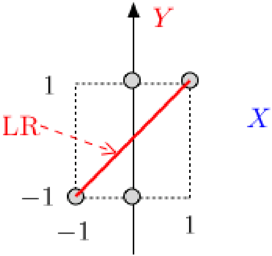

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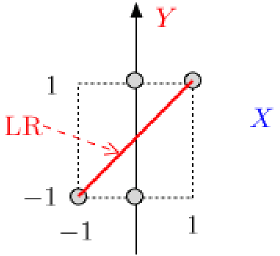

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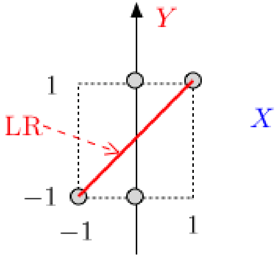

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Linear Regression Examples

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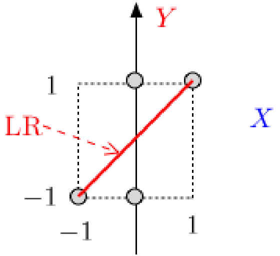

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Linear Regression Examples

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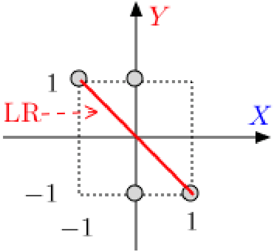

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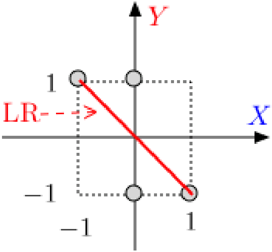

We find:

$E[X] =$

Linear Regression Examples

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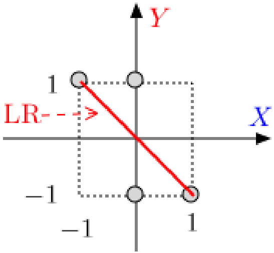

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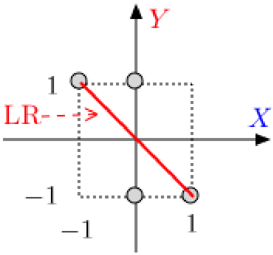

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$$E[X] = 0; E[Y] =$$

Linear Regression Examples

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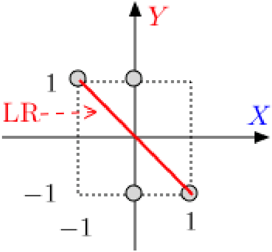

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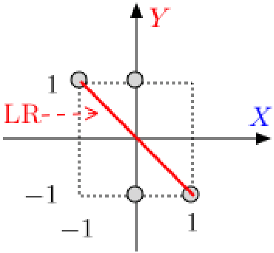

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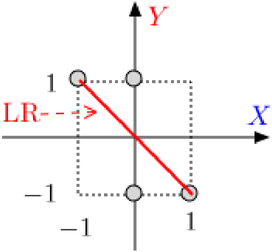

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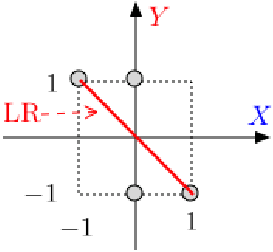

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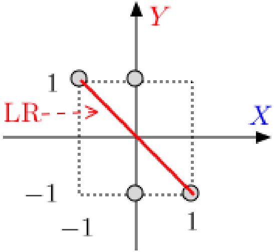

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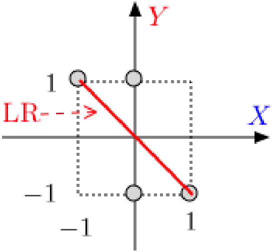

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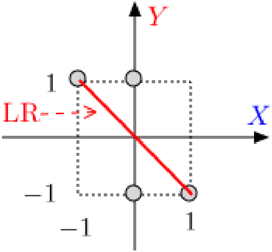

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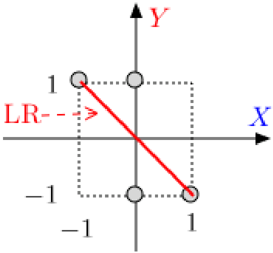

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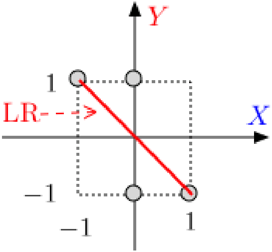

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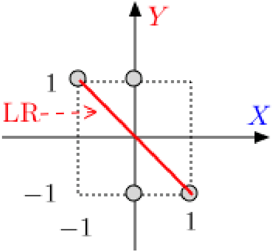

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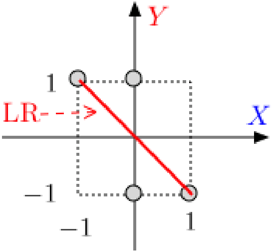

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Linear Regression Examples

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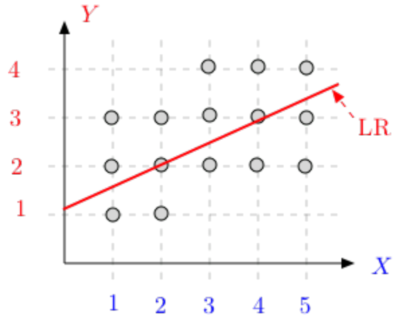

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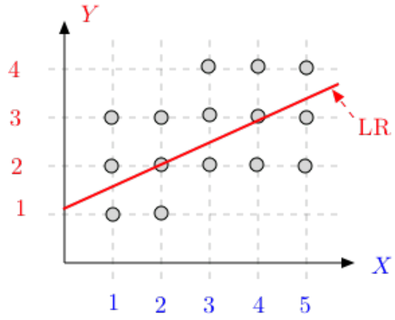

We find:

$$E[X] =$$

Linear Regression Examples

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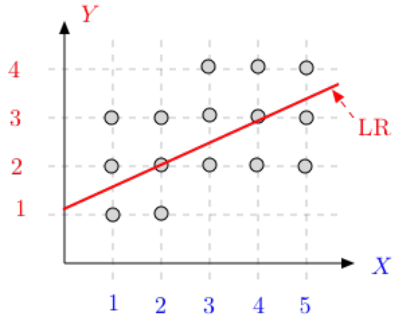

We find:

$E[X] = 3;$

Linear Regression Examples

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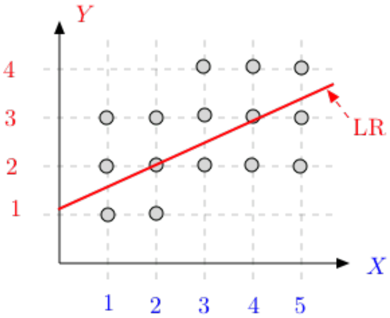

We find:

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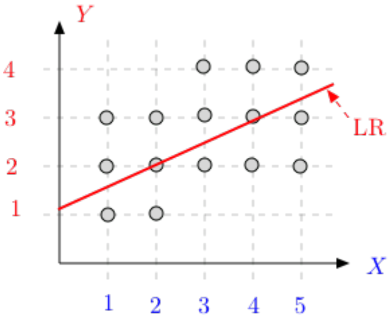

We find:

$$E[X] = 3; E[Y] = 2.5;$$

Linear Regression Examples

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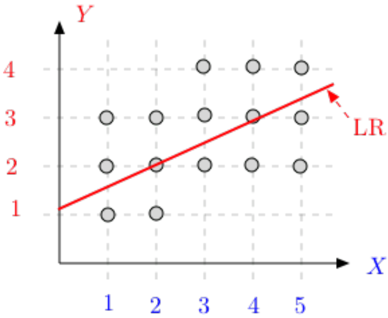

We find:

$$E[X] = 3; E[Y] = 2.5; E[X^2] = (3/15)(1 + 2^2 + 3^2 + 4^2 + 5^2) = 11;$$

Linear Regression Examples

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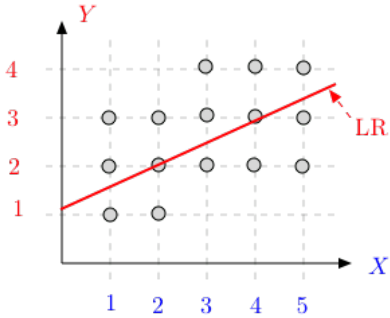

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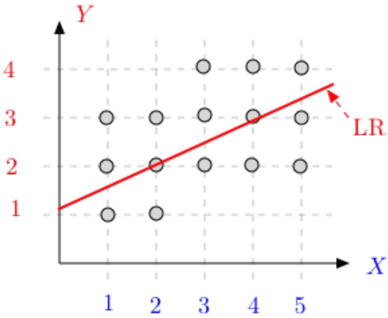

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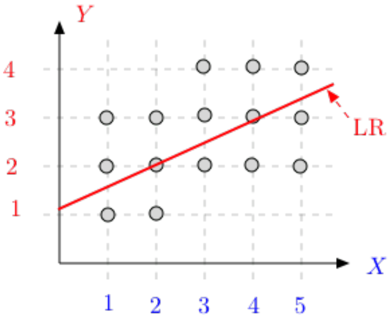

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$$var[X] = 11 - 9 = 2; cov(X, Y) = 8.4 - 3 \times 2.5 = 0.9;$$

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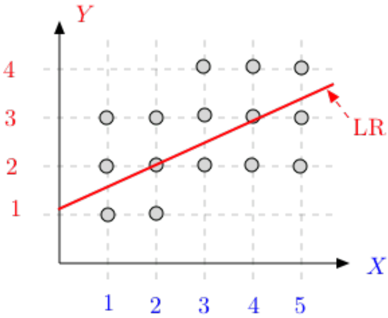

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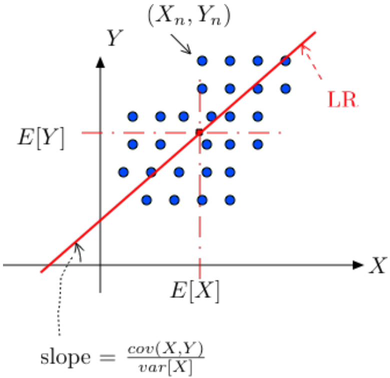
$$var[X] = 11 - 9 = 2; cov(X, Y) = 8.4 - 3 \times 2.5 = 0.9;$$
$$\text{LR: } \hat{Y} = 2.5 + \frac{0.9}{2}(X - 3) = 1.15 + 0.45X.$$

LR: Another Figure



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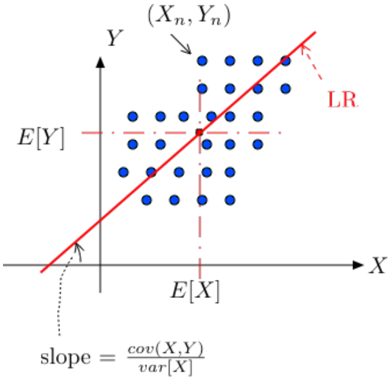

Note that

- the LR line goes through $(E[X], E[Y])$

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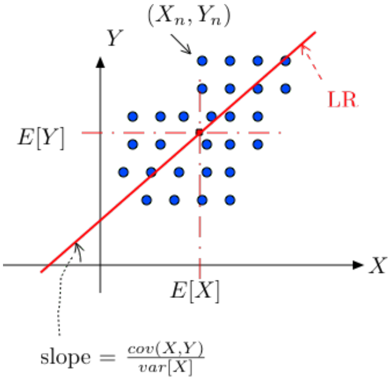
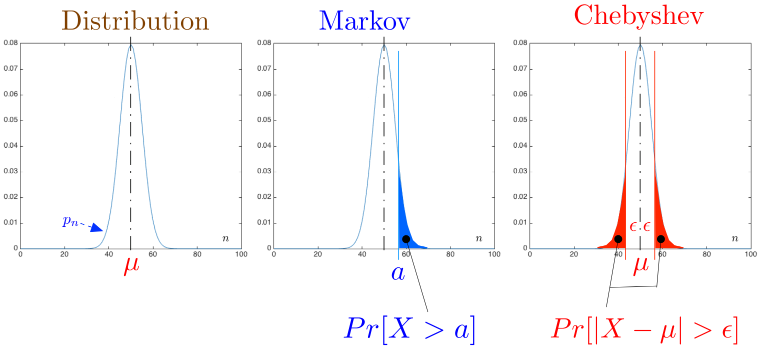


Note that

- the LR line goes through $(E[X], E[Y])$
- its slope is $\frac{cov(X,Y)}{var(X)}$.

Inequalities: An Overview



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Lake Woebegone: Poll

What is true?

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(A) Everyone is above average (on midterm)

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(A) Everyone is above average (on midterm)
False. Average would be higher.
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False. Consider $Pr[X = -2] = 1/3$ and $Pr[X = 1] = 2/3$. $E[X] = 0$.

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False. Average would be higher.
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(C) For the midterm with no negative scores, at most half the people can be more than twice the average.

Lake Woebegone: Poll

What is true?

(A) Everyone is above average (on midterm)
False. Average would be higher.
(B) For a random variable, at most half the people can be more than twice the average.
False. Consider $Pr[X=-2]=1/3$ and $Pr[X=1]=2/3$. $E[X]=0$.
(C) For the midterm with no negative scores, at most half the people can be more than twice the average.
True. Otherwise average would be higher.

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What is true?

(A) Everyone is above average (on midterm)
False. Average would be higher.
(B) For a random variable, at most half the people can be more than twice the average.
False. Consider $Pr[X = -2] = 1/3$ and $Pr[X = 1] = 2/3$. $E[X] = 0$.
(C) For the midterm with no negative scores, at most half the people can be more than twice the average.
True. Otherwise average would be higher.

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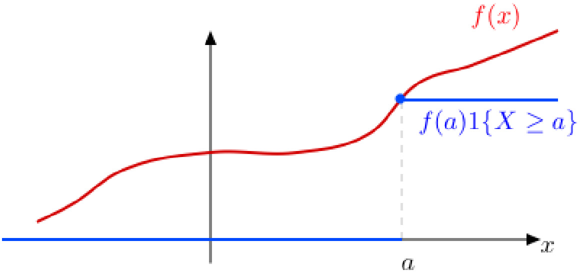
That is, $\sum_v Pr[X = v]1\{v \geq a\} \leq \sum_v Pr[X = v]\frac{f(v)}{f(a)}$.

□

Intuition: $E[f(X)] \geq f(a)Pr[X > a] = f(a)Pr[X > f(a)]$.

A picture



$$f(a)1\{X \geq a\} \leq f(x) \Rightarrow 1\{X \geq a\} \leq \frac{f(X)}{f(a)}$$

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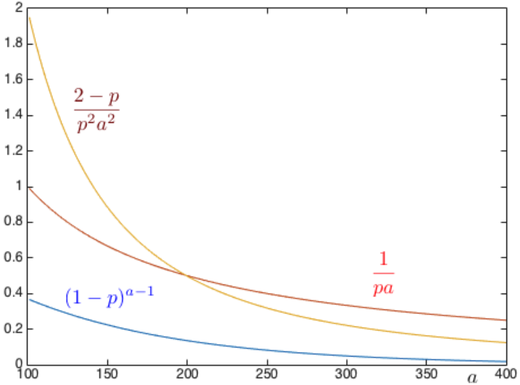

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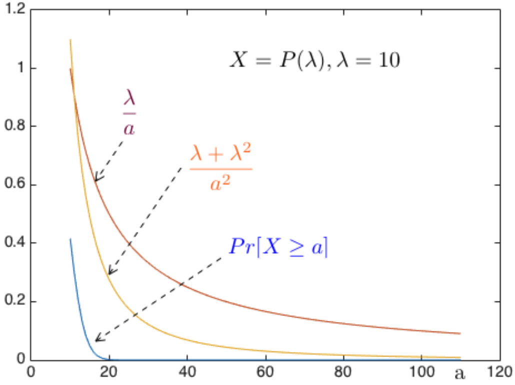

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This result confirms that the variance measures the "deviations from the mean."

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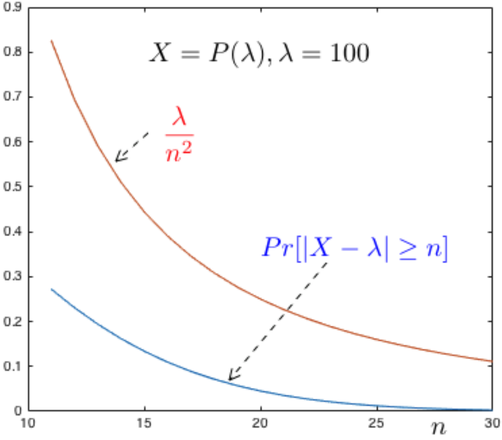

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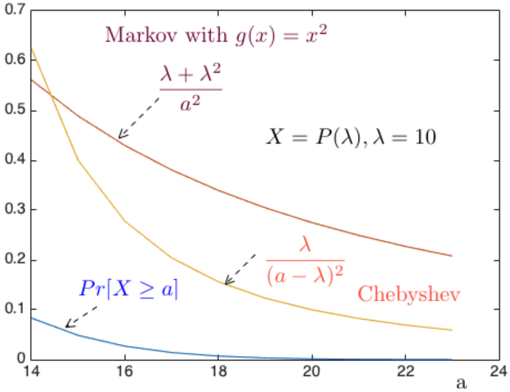

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Fraction of H's

Here is a classical application of Chebyshev's inequality.

How likely is it that the fraction of H's differs from 50%?

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$$Y_n = \frac{X_1 + \cdots + X_n}{n}, \text{ for } n \geq 1.$$

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We look at a general case next.

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  - What surgeon do you choose?

Coin Flips: Intuition

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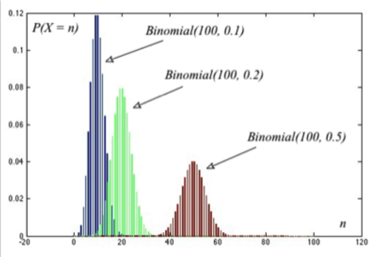

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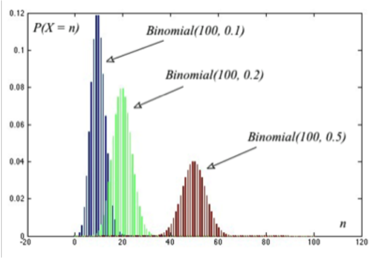

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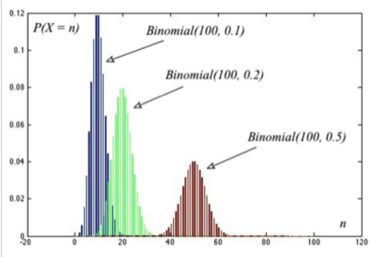

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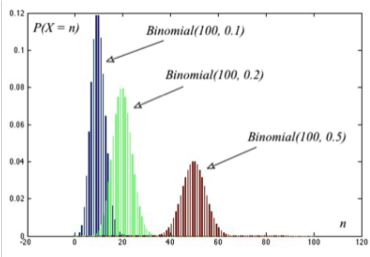

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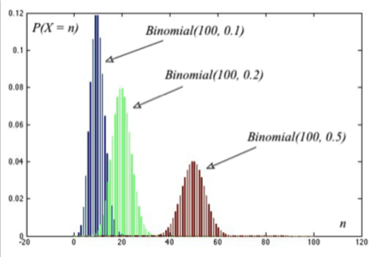

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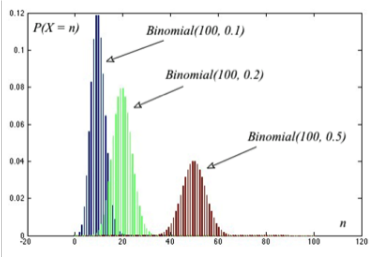

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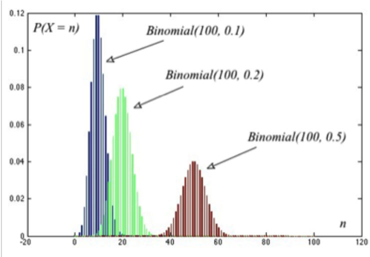

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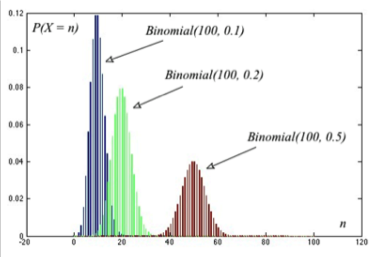

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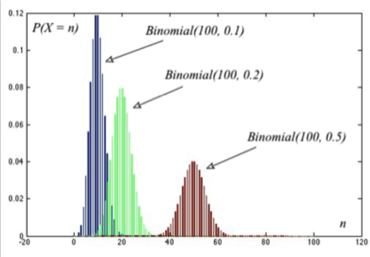

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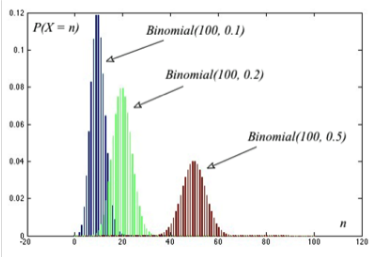

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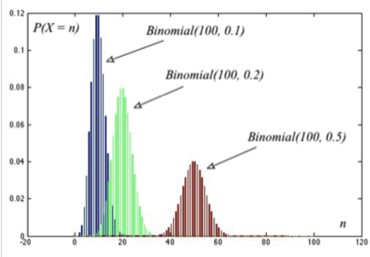

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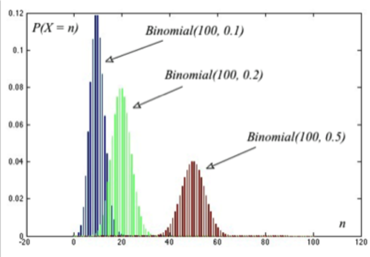

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$$Pr[|A_n - \mu| \geq 4.5\sigma/\sqrt{n}] \leq \frac{var(A_n)}{[4.5\sigma/\sqrt{n}]^2} = \frac{n}{20\sigma^2} var(A_n).$$

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Confidence Interval: Analysis

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$\square$

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An illustration:

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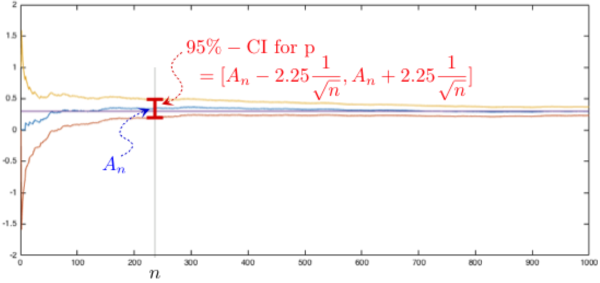

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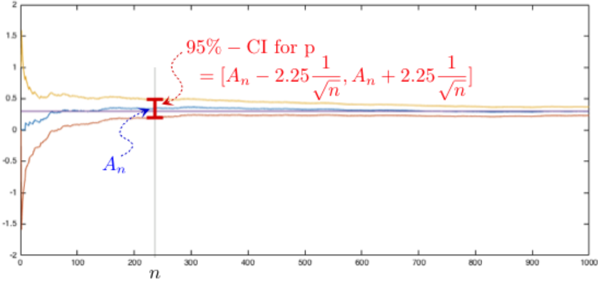

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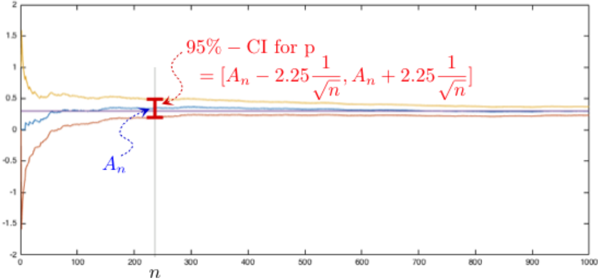

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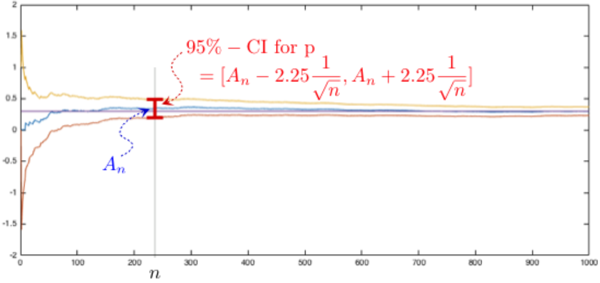

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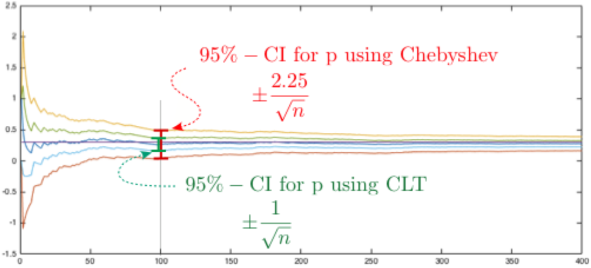

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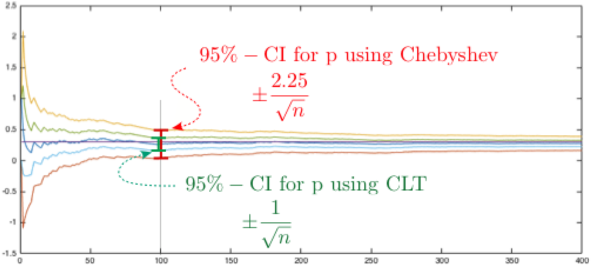

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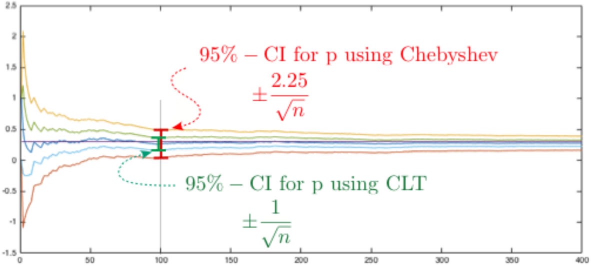

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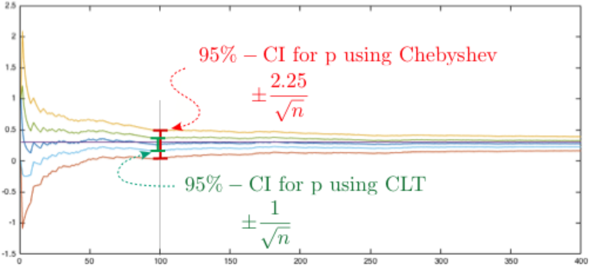

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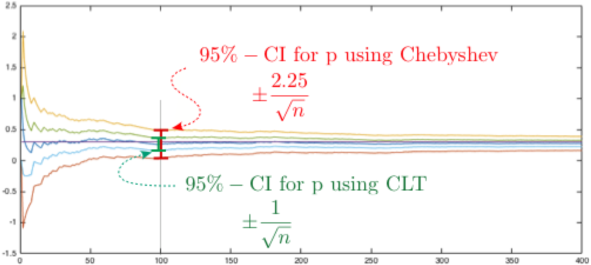

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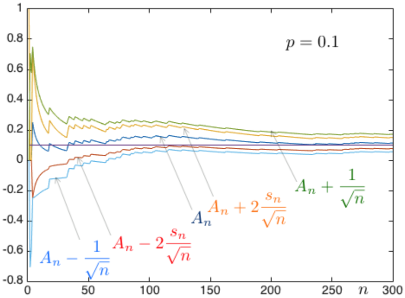

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6. Examples: $B(p), G(p)$, which coin is better?
7. In some cases, one can replace σ by the empirical standard deviation.