Lecture 7. Outline.

1. Modular Arithmetic.
   Clock Math!!!
2. Inverses for Modular Arithmetic: Greatest Common Divisor.
   Division!!!
3. Euclid's GCD Algorithm.
   A little tricky here!

Next Up.

Modular Arithmetic.

Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours? 6:00!
What time is it in 15 hours? 16:00!
Actually 4:00.

16 is the "same as 4" with respect to a 12 hour clock system.
Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.
$101 = 12 \times 8 + 5$.
5 is the same as 101 for a 12 hour clock system.
Clock time equivalent up to addition of any integer multiple of 12.

Custom is only to use the representative in $\{12, 1, \ldots, 11\}$
(Almost remainder, except for 12 and 0 are equivalent.)

Day of the week.

This is Tuesday is February 11, 2024.
What day is it a year from now? on February, 2025?
Number days.
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.
5 days from then. day 7 or day 0 or Sunday.
25 days from then. day 27 or day 3. $27 = (7)3 + 3$
two days are equivalent up to addition/subtraction of multiple of 7.
11 days from then is day 13 or day 6 which is Saturday!

What day is it a year from then?
This year is not a leap year. So 365 days from now.
Day 2+366 or day 368. Leap year.
Smallest representation:
subtract 7 until smaller than 7.
divide and get remainder.
370/7 leaves quotient of 52 and remainder 6. $369 = 7(52) + 6$
or September 18, 2025 is a Saturday.

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 2.
It is day $2 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 4.
Get Day: $2 + 2 \times 20 + 1 \times 60 = 102$
Remainder when dividing by 7? $102 = 14 \times 7 + 4$.
Or February, 2105 is Thursday!

Further Simplify Calculation:
20 has remainder 6 when divided by 7.
60 has remainder 4 when divided by 7.
Get Day: $2 + 2 \times 6 + 1 \times 4 = 18$.
Or Day 4. February 11, 2105 is Thursday.

"Reduce" at any time in calculation!

Modular Arithmetic: refresher.

x **is congruent to** y **modulo** m or "$x \equiv y \pmod{m}$"
if and only if $(x - y)$ is divisible by m.
...or x and y have the same remainder w.r.t. m.
...or $x = y + km$ for some integer k.

Mod 7 equivalence or *residue* classes:
$\{\ldots, -7, 0, 7, 14, \ldots\}$ $\{\ldots, -6, 1, 8, 15, \ldots\}$ ...

Useful Fact: Addition, subtraction, multiplication can be done with any equivalent x and y.

or " $a \equiv c \pmod{m}$ and $b \equiv d \pmod{m}$
$\implies a + b \equiv c + d \pmod{m}$ and $a \cdot b = c \cdot d \pmod{m}$"

Proof: If $a \equiv c \pmod{m}$, then $a = c + km$ for some integer k.
If $b \equiv d \pmod{m}$, then $b = d + jm$ for some integer j.
Therefore, $a + b = c + d + (k + j)m$ and since $k + j$ is integer.
$\implies a + b \equiv c + d \pmod{m}$. □

Can calculate with representative in $\{0, \ldots, m - 1\}$.

Notation

$x \pmod m$ or $\mod(x,m)$
- remainder of x divided by m in $\{0,\ldots,m-1\}$.

$\mod(x,m) = x - \lfloor \frac{x}{m} \rfloor m$

$\lfloor \frac{x}{m} \rfloor$ is quotient.

$\mod(29,12) = 29 - (\lfloor \frac{29}{12} \rfloor) \times 12 = 29 - (2) \times 12 = \cancel{4} = 5$

Work in this system.
$a \equiv b \pmod m$.
Says two integers a and b are equivalent modulo m.

Modulus is m

$6 \equiv 3+3 \equiv 3+10 \pmod 7$.

$6 = 3+3 = 3+10 \pmod 7$.

Generally, not
not $6 \pmod 7 = 13 \pmod 7$.
But probably won't take off points, still..not what is happening.

Inverses and Factors.

Division: multiply by multiplicative inverse.

$$2x = 3 \implies (\frac{1}{2}) \cdot 2x = (\frac{1}{2}) \cdot 3 \implies x = \frac{3}{2}.$$

Multiplicative inverse of x is y where $xy = 1$;
1 ***is multiplicative identity element.***

In modular arithmetic, 1 is the multiplicative identity element.

Multiplicative inverse of x **mod** m is y with $xy = 1 \pmod m$.

For 4 modulo 7 inverse is 2: $2 \cdot 4 \equiv 8 \equiv 1 \pmod 7$.

Can solve $4x = 5 \pmod 7$.
$2(4x) = 2 \cdot 5 \pmod 7$
$8x = 10 \pmod 7$
$x = 3 \pmod 7$
Check! $4(3) = 12 = 5 \pmod 7$.

$x = 3 \pmod 7$ ::: Check! $4(3) = 12 = 5 \pmod 7$.

For 8 modulo 12: no multiplicative inverse!

"Common factor of 4" $\implies$
$8k - 12\ell$ is a multiple of four for any ℓ and $k \implies 8k - 12\ell \neq 1 \implies$
$8k \not\equiv 1 \pmod{12}$ for any k.

Poll

Mark true statements.
(A) Mutliplicative inverse of 2 mod 5 is 3 mod 5.
(B) The multiplicative inverse of $((n-1) \pmod n) = ((n-1) \pmod n)$.
(C) Multiplicative inverse of 2 mod 5 is 0.5.
(D) Multiplicative inverse of $4 = -1 \pmod 5$.
(E) $(-1)x(-1) = 1$. Woohoo.
(F) Multiplicative inverse of 4 mod 5 is 4 mod 5.

(C) is false. 0.5 has no meaning in arithmetic modulo 5.

Greatest Common Divisor and Inverses.

Thm:
If greatest common divisor of x and m, $\gcd(x,m)$, is 1, then x has a multiplicative inverse modulo m.

Proof $\implies$**:**
Claim: The set $S = \{0x, 1x, \ldots, (m-1)x\}$ contains $y \equiv 1 \mod m$ if all distinct modulo m.

Each of m numbers in S correspond to one of m equivalence classes modulo m.
$\implies$ One must correspond to 1 modulo m. Inverse Exists!

Proof of Claim: If not distinct, then $\exists a, b \in \{0,\ldots,m-1\}$, $a \neq b$, where
$(ax \equiv bx \pmod m) \implies (a-b)x \equiv 0 \pmod m$
Or $(a-b)x = km$ for some integer k.

$gcd(x,m) = 1$
$\implies$ Prime factorization of m and x do not contain common primes.
$\implies$ $(a-b)$ factorization contains all primes in m's factorization.
So $(a-b)$ has to be multiple of m.
$\implies (a-b) \geq m$. But $a, b \in \{0, \ldots m-1\}$. Contradiction. □

Proof review. Consequence.

Thm: If $\gcd(x,m) = 1$, then x has a multiplicative inverse modulo m.

Proof Sketch: The set $S = \{0x, 1x, \ldots, (m-1)x\}$ contains $y \equiv 1 \mod m$ if all distinct modulo m.
... □
For $x = 4$ and $m = 6$. All products of 4...
$S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}$
reducing $\pmod 6$
$S = \{0, 4, 2, 0, 4, 2\}$
Not distinct. Common factor 2. Can't be 1. No inverse.

For $x = 5$ and $m = 6$.
$S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\}$
All distinct, contains 1! 5 is multiplicative inverse of 5 $\pmod 6$.
(Hmm. What number is it own multiplicative inverse?) 1 and -1.

$5x = 3 \pmod 6$ What is x? Multiply both sides by 5.
$x = 15 = 3 \pmod 6$

$4x = 3 \pmod 6$ No solutions. Can't get an odd.
$4x = 2 \pmod 6$ Two solutions! $x = 2, 5 \pmod 6$

Very different for elements with inverses.

Proof Review 2: Bijections.

If gcd(x,m) = 1.
Then the function $f(a) = xa \mod m$ is a bijection.
One to one: there is a unique pre-image(single x where $y = f(x)$.)
Onto: the sizes of the domain and co-domain are the same.

$x = 3, m = 4$.
$f(1) = 3(1) = 3 \pmod 4$,
$f(2) = 6 = 2 \pmod 4$,
$f(3) = 1 \pmod 3$.
Oh yeah. $f(0) = 0 \pmod 3$.

Bijection $\equiv$ unique pre-image and same size.
All the images are distinct. $\implies$ unique pre-image for any image.

$x = 2, m = 4$.
$f(1) = 2$,
$f(2) = 0$,
$f(3) = 2$
Oh yeah. $f(0) = 0$.

Not a bijection.

Poll

Which is bijection?
(A) $f(x) = x$ for domain and range being $\mathbb{R}$
(B) $f(x) = ax \pmod n$ for $x \in \{0,...,n-1\}$ and $gcd(a,n) = 2$
(C) $f(x) = ax \pmod n$ for $x \in \{0,...,n-1\}$ and $gcd(a,n) = 1$

(B) is not. maps from $\{0,\ldots,n-1\}$ to $\{0,2,\ldots,n-2\}$.

In general, image consists to multiples of $gcd(a,n)$.

Only if

Thm: If $gcd(x,m) = d \neq 1$ then x has no multiplicative inverse modulo m.

Assume the inverse of a is x^{-1}, or $ax = 1 + km$.

$x = nd$ and $m = \ell d$ for $d > 1$.

Thus,

$a(nd) = 1 + k\ell d$ or

$d(na - k\ell) = 1$.

But $d > 1$ and $z = (na - k\ell) \in \mathbb{Z}$.

so $dz \neq 1$ and $dz = 1$. Contradiction.

□

Finding inverses.

How to find the inverse?

How to find **if** x has an inverse modulo m?

Find gcd (x,m).
Greater than 1? No multiplicative inverse.
Equal to 1? Multiplicative inverse.

Algorithm: Try all numbers up to x to see if it divides both x and m.

Very slow.

Inverses

Next up.

Euclid's Algorithm.
Runtime.
Euclid's Extended Algorithm.

Refresh

Does 2 have an inverse mod 8? No.
Any multiple of 2 is 2 away from $0 + 8k$ for any $k \in \mathbb{N}$.

Does 2 have an inverse mod 9? Yes. 5
$2(5) = 10 = 1 \mod 9$.

Does 6 have an inverse mod 9? No.
Any multiple of 6 is 3 away from $0 + 9k$ for any $k \in \mathbb{N}$.
$3 = gcd(6,9)$!

x has an inverse modulo m if and only if
$gcd(x,m) > 1$? No.
$gcd(x,m) = 1$? Yes.

Now what?:
Compute gcd!
Compute Inverse modulo m.

Divisibility...

Notation: $d|x$ means "d divides x" or
$x = kd$ for some integer k.

Fact: If $d|x$ and $d|y$ then $d|(x+y)$ and $d|(x-y)$.

Is it a fact? Yes? No?

Proof: $d|x$ and $d|y$ or
$x = \ell d$ and $y = kd$

$\implies x - y = kd - \ell d = (k - \ell)d \implies d|(x-y)$

□

More divisibility

Notation: $d|x$ means "d divides x" or
$x = kd$ for some integer k.

Lemma 1: If $d|x$ and $d|y$ then $d|y$ and $d| \mod(x,y)$.

Proof:

$$\begin{aligned} \mod(x,y) &= x - \lfloor x/y \rfloor \cdot y \\ &= x - s \cdot y \quad \text{for integer } s \\ &= kd - s\ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d \\ &= (k - s\ell)d \end{aligned}$$

Therefore $d| \mod(x,y)$. And $d|y$ since it is in condition. □

Lemma 2: If $d|y$ and $d| \mod(x,y)$ then $d|y$ and $d|x$.
Proof...: Similar. Try this at home. □ish.

GCD Mod Corollary: $\gcd(x,y) = \gcd(y, \mod(x,y))$.
Proof: x and y have **same** set of common divisors as x and $\mod(x,y)$ by Lemma 1 and 2.
Same common divisors $\implies$ largest is the same. □

Euclid's algorithm.

GCD Mod Corollary: $\gcd(x,y) = \gcd(y, \mod(x,y))$.

Hey, what's $\gcd(7,0)$? 7 since 7 divides 7 and 7 divides 0
What's $\gcd(x,0)$? x

```
(define (euclid x y)
  (if (= y 0)
    x
    (euclid  y (mod x y))))  ***
```

Theorem: (euclid x y) $= \gcd(x,y)$ if $x \geq y$.

Proof: Use Strong Induction.
Base Case: $y = 0$, "x divides y and x"
$\implies$ "x is common divisor and clearly largest."
Induction Step: $\mod(x,y) < y \leq x$ when $x \geq y$

call in line (***) meets conditions plus arguments "smaller"
and by strong induction hypothesis
computes $\gcd(y, \mod(x,y))$
which is $\gcd(x,y)$ by GCD Mod Corollary. □

Excursion: Value and Size.

Before discussing running time of gcd procedure...

What is the value of 1,000,000?

one million or 1,000,000!

What is the "size" of 1,000,000?

Number of digits in base 10: 7.

Number of bits (a digit in base 2): 21.

For a number x, what is its size in bits?

$$n = b(x) \approx \log_2 x$$

Euclid procedure is fast.

Theorem: (euclid x y) uses $2n$ "divisions" where $n = b(x) \approx \log_2 x$.

Is this good? Better than trying all numbers in $\{2, \dots y/2\}$?

Check 2, check 3, check 4, check 5 ..., check $y/2$.

If $y \approx x$ roughly y uses n bits ...
2^{n-1} divisions! Exponential dependence on size!

101 bit number. $2^{100} \approx 10^{30} =$ "million, trillion, trillion" divisions!

$2n$ is much faster! .. roughly 200 divisions.

Poll.

Assume $\log_2 1{,}000{,}000$ is 20 to the nearest integer.
Mark what's true.

(A) The size of 1,000,000 is 20 bits.
(B) The size of 1,000,000 is one million.
(C) The value of 1,000,000 is one million.
(D) The value of 1,000,000 is 20.

(A) and (C).

Poll

Which are correct?

(A) gcd(700,568) = gcd (568,132)
(B) gcd(8,3) = gcd(3,2)
(C) gcd(8,3) = 1
(D) gcd(4,0) = 4

Algorithms at work.

Trying everything

Check 2, check 3, check 4, check 5 . . . , check $y/2$.

"(gcd x y)" at work.

```
euclid(700,568)
  euclid(568, 132)
    euclid(132, 40)
      euclid(40, 12)
        euclid(12, 4)
          euclid(4, 0)
            4
```

Notice: The first argument decreases rapidly.
At least a factor of 2 in two recursive calls.

(The second is less than the first.)

Runtime Proof.

```
(define (euclid x y)
 (if (= y 0)
     x
    (euclid  y (mod x y))))
```

Theorem: (euclid x y) uses $O(n)$ "divisions" where $n = b(x)$.

Proof:

Fact:
First arg decreases by at least factor of two in two recursive calls.

After $2\log_2 x = O(n)$ recursive calls, argument x is 1 bit number.
One more recursive call to finish.
1 division per recursive call.
$O(n)$ divisions.

□

Runtime Proof (continued.)

```
(define (euclid x y)
 (if (= y 0)
     x
    (euclid  y (mod x y))))
```

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 1: $y < x/2$, first argument is y
$\implies$ true in one recursive call;

Case 2: Will show "$y \geq x/2$" $\implies$ "$mod(x,y) \leq x/2$."

mod (x,y) is second argument in next recursive call,
and becomes the first argument in the next one.

When $y \geq x/2$, then

$\lfloor \frac{x}{y} \rfloor = 1$,

mod $(x,y) = x - y\lfloor \frac{x}{y} \rfloor = x - y \leq x - x/2 = x/2$

□

v

Finding an inverse?

We showed how to efficiently tell if there is an inverse.

Extend euclid to find inverse.

Euclid's GCD algorithm.

```
(define (euclid x y)
 (if (= y 0)
     x
    (euclid  y (mod x y))))
```

Computes the gcd(x,y) in $O(n)$ divisions.

For x and m, if gcd$(x,m) = 1$ then x has an inverse modulo m.

Multiplicative Inverse.

GCD algorithm used to tell **if** there is a multiplicative inverse.

How do we **find** a multiplicative inverse?

Tuesday

Modular Arithmetic Lecture in a minute.

Modular Arithmetic: $x \equiv y \pmod{N}$ if $x = y + kN$ for some integer k.

For $a \equiv b \pmod{N}$, and $c \equiv d \pmod{N}$,
$ac = bd \pmod{N}$ and $a + b = c + d \pmod{N}$.

Division? Multiply by multiplicative inverse.
$a \pmod{N}$ has multiplicative inverse, $a^{-1} \pmod{N}$.
If and only if $gcd(a, N) = 1$.

Why? If: $f(x) = ax \pmod{N}$ is a bijection on $\{1, \ldots, N-1\}$.
$ax - ay = 0 \pmod{N} \implies a(x-y)$ is a multiple of N.
If $gcd(a, N) = 1$,
then $(x - y)$ must contain all primes in prime factorization of N,
and is therefore be bigger than N.
Only if: For $a = xd$ and $N = yd$,
any $ma + kN = d(mx - ky)$ or is a multiple of d,
and is not 1.

Euclid's Alg: $gcd(x, y) = gcd(y \mod x, x)$
Fast cuz value drops by a factor of two every two recursive calls.

Know if there is an inverse, but efficiently find it? On Thursday!