Today

Finish Euclid.

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Bijection/CRT/Isomorphism.

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Fermat's Little Theorem.

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Quick review

Review runtime proof.

Runtime Proof.

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(define (euclid x y)
 (if (= y 0)
     x
    (euclid  y (mod x y))))
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Theorem: (euclid x y) uses $O(n)$ "divisions" where $n = b(x)$.

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One more recursive call to finish.

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$O(n)$ divisions.

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mod (x, y) is second argument in next recursive call,
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Mark correct answers.
Note: Mod(x,y) is the remainder of x divided by y and $y < x$.

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(A) $\mod(x, y) < y$
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(B) If euclid(x,y) calls euclid(u,v) calls euclid (a,b) then $a <= x/2$.
(C) euclid(x,y) calls euclid (u,v) means u = y.

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Finding an inverse?

We showed how to efficiently tell if there is an inverse.

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Extend euclid to find inverse.

Euclid's GCD algorithm.

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For x and m, if $\gcd(x, m) = 1$ then x has an inverse modulo m.

Multiplicative Inverse.

GCD algorithm used to tell **if** there is a multiplicative inverse.

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How do we **find** a multiplicative inverse?

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Check: 3(12)

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Check: $3(12) = 36 = 1 \pmod{35}$.

Make d out of multiples of x and y..?

```
gcd(35,12)
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Make *d* out of multiples of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
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Make d out of multiples of x and y..?

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gcd(35,12)
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Make *d* out of multiples of *x* and *y*..?

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gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
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How did gcd get 11 from 35 and 12?

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$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

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How does gcd get 1 from 12 and 11?

Make d out of multiples of x and y..?

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$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

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$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

Make d out of multiples of x and y..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Make d out of multiples of x and y..?

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gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
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Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.

Make d out of multiples of x and y..?

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gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
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$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$1 = 12 - (1)11$

Make d out of multiples of x and y..?

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gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

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But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$1 = 12 - (1)11 = 12 - (1)(35 - (2)12)$
Get 11 from 35 and 12 and plugin....

Make d out of multiples of x and y..?

```
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But we want 1 from sum of multiples of 35 and 12?

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$1 = 12 - (1)11 = 12 - (1)(35 - (2)12) = (3)12 + (-1)35$
Get 11 from 35 and 12 and plugin.... Simplify.

Make d out of multiples of x and y..?

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Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$1 = 12 - (1)11 = 12 - (1)(35 - (2)12) = (3)12 + (-1)35$
Get 11 from 35 and 12 and plugin.... Simplify. $a = 3$ and $b = -1$.

Extended GCD Algorithm.

```
ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
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Claim: Returns (d,a,b): $d = gcd(x,y)$ and $d = ax + by$.

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      ext-gcd(1,0)
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Claim: Returns (d,a,b): $d = gcd(x,y)$ and $d = ax + by$.
Example: $a - \lfloor x/y \rfloor \cdot b =$

```
ext-gcd(35,12)
  ext-gcd(12, 11)
    ext-gcd(11, 1)
      ext-gcd(1,0)
      return (1,1,0) ;; 1 = (1)1 + (0) 0
```

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ext-gcd(35,12)
  ext-gcd(12, 11)
    ext-gcd(11, 1)
      ext-gcd(1,0)
      return (1,1,0) ;; 1 = (1)1 + (0) 0
    return (1,0,1)   ;; 1 = (0)11 + (1)1
```

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  if y = 0 then return(x, 1, 0)
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```

Claim: Returns (d,a,b): $d = gcd(x,y)$ and $d = ax + by$.
Example: $a - \lfloor x/y \rfloor \cdot b = \quad 0 - \lfloor 12/11 \rfloor \cdot 1 = -1$

```
ext-gcd(35,12)
  ext-gcd(12, 11)
    ext-gcd(11, 1)
      ext-gcd(1,0)
      return (1,1,0) ;; 1 = (1)1 + (0) 0
    return (1,0,1)   ;; 1 = (0)11 + (1)1
  return (1,1,-1)    ;; 1 = (1)12 + (-1)11
```

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ext-gcd(x,y)
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```

Claim: Returns (d, a, b): $d = gcd(x, y)$ and $d = ax + by$.
Example: $a - \lfloor x/y \rfloor \cdot b =$ $1 - \lfloor 35/12 \rfloor \cdot (-1) = 3$

```
    ext-gcd(35,12)
      ext-gcd(12, 11)
        ext-gcd(11, 1)
          ext-gcd(1,0)
          return (1,1,0) ;; 1 = (1)1 + (0) 0
        return (1,0,1)   ;; 1 = (0)11 + (1)1
      return (1,1,-1)    ;; 1 = (1)12 + (-1)11
   return (1,-1, 3)      ;; 1 = (-1)35 +(3)12
```

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    ext-gcd(35,12)
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        return (1,0,1)   ;; 1 = (0)11 + (1)1
      return (1,1,-1)    ;; 1 = (1)12 + (-1)11
   return (1,-1, 3)      ;; 1 = (-1)35 +(3)12
```

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         return (d, b, a - floor(x/y)  * b)
```

Theorem: Returns (d, a, b), where $d = gcd(a, b)$ and

$$d = ax + by.$$

Correctness.

Proof: Strong Induction.[1]

[1] Assume d is $gcd(x, y)$ by previous proof.

Correctness.

Proof: Strong Induction.[1]
Base: ext-gcd$(x,0)$ returns $(d = x, 1, 0)$ with $x = (1)x + (0)y$.

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$$\begin{aligned} d &= ay + b\cdot(\mod(x,y)) \\ &= ay + b\cdot(x - \lfloor\frac{x}{y}\rfloor y) \end{aligned}$$

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$$\begin{aligned} d &= ay + b \cdot (\mod(x, y)) \\ &= ay + b \cdot (x - \lfloor \frac{x}{y} \rfloor y) \\ &= bx + (a - \lfloor \frac{x}{y} \rfloor \cdot b)y \end{aligned}$$

[1] Assume d is $gcd(x, y)$ by previous proof.

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Proof: Strong Induction.[1]
Base: ext-gcd$(x,0)$ returns $(d=x,1,0)$ with $x=(1)x+(0)y$.

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And ext-gcd returns $(d, b, (a - \lfloor\frac{x}{y}\rfloor\cdot b))$ so theorem holds!

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$$\begin{aligned} d &= ay+b\cdot(\mod(x,y)) \\ &= ay+b\cdot(x-\lfloor\frac{x}{y}\rfloor y) \\ &= bx+(a-\lfloor\frac{x}{y}\rfloor\cdot b)y \end{aligned}$$

And ext-gcd returns $(d,b,(a-\lfloor\frac{x}{y}\rfloor\cdot b))$ so theorem holds! $\square$

[1] Assume d is $gcd(x,y)$ by previous proof.

Review Proof: step.

```
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Recursively: $d = ay + b(x - \lfloor \frac{x}{y} \rfloor \cdot y)$

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Recursively: $d = ay + b(x - \lfloor \frac{x}{y} \rfloor \cdot y) \implies d = bx - (a - \lfloor \frac{x}{y} \rfloor b)y$

Returns $(d, b, (a - \lfloor \frac{x}{y} \rfloor \cdot b))$.

Hand Calculation Method for Inverses.

Example: $\gcd(7,60) = 1$.

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$$\begin{aligned} 7(0) + 60(1) &= 60 \\ 7(1) + 60(0) &= 7 \end{aligned}$$

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Example: $\gcd(7,60) = 1$.
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$$\begin{aligned} 7(0) + 60(1) &= 60 \\ 7(1) + 60(0) &= 7 \\ 7(-8) + 60(1) &= 4 \end{aligned}$$

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Confirm: $-119+120=1$

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Confirm: $-119 + 120 = 1$

Note: an “iterative” version of the e-gcd algorithm.

Fundamental Theorem of Arithmetic.

Thm: Every natural number can be written as the product of primes.

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Generalization: things with a “division algorithm”.

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Generalization: things with a "division algorithm".

One example: polynomial division.

No shared common factors, and products.

Claim: For $x, y, z \in \mathbb{Z}^+$ with $gcd(x, y) = 1$ and $x|yz$ then $x|z$.

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Observe: $x|axz$ and $x|byz$ (since $x|yz$), and x divides the sum.

No shared common factors, and products.

Claim: For $x, y, z \in \mathbb{Z}^+$ with $gcd(x, y) = 1$ and $x|yz$ then $x|z$.

Idea(restatemten): x doesn't share common factors with y
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Wrap-up

Conclusion: Can find multiplicative inverses in $O(n)$ time!

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Internet Security: Soon.

Lots of Mods

$x = 5 \pmod 7$ and $x = 3 \pmod 5$.

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What is $x \pmod{35}$?

Let's try 5.

Lots of Mods

$x = 5 \pmod{7}$ and $x = 3 \pmod{5}$.

What is $x \pmod{35}$?

Let's try 5. Not $3 \pmod{5}$!

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If $x = 5 \pmod 7$

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Let's try 5. Not $3 \pmod{5}$!
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If $x = 5 \pmod{7}$
then x is in $\{5, 12, 19, 26, 33\}$.

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Oh, only 33 is $3 \pmod 5$.

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A bit slow for large values.

Simple Chinese Remainder Theorem.

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My love is won.

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My love is won. Zero and One.

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Find $x = a \pmod{m}$ and $x = b \pmod{n}$

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Find $x = a \pmod{m}$ and $x = b \pmod{n}$ where gcd(m, n)=1.

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CRT Thm: There is a unique solution $x \pmod{mn}$.

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Proof (solution exists):

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Consider $u = n(n^{-1} \pmod{m})$.

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Let $x = au + bv$.

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$v = 1 \pmod{n}$ $\quad v = 0 \pmod{m}$
Let $x = au + bv$.
$x = a \pmod{m}$ since $bv = 0 \pmod{m}$ and $au = a \pmod{m}$

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$x = a \pmod{m}$ since $bv = 0 \pmod{m}$ and $au = a \pmod{m}$
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Proof (solution exists):
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Consider $v = m(m^{-1} \pmod{n})$.
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Let $x = au + bv$.
$x = a \pmod{m}$ since $bv = 0 \pmod{m}$ and $au = a \pmod{m}$
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Thus there is a solution.

Simple Chinese Remainder Theorem.

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Basis of hardware accelerators for security.

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For a prime modulus, we can reduce exponents modulo $p-1$!

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Unique prime factorization of any natural number.

Claim: if $p|n$ and $n = xy$, $p|x$ of $p|x$.

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Product of elts == for range/domain: a^{p-1} factor in range.